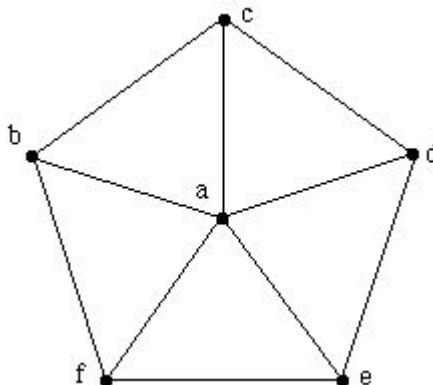




p.629, icon at Example 14

#1. Use powers of the adjacency matrix to find the following numbers of paths in this graph:

- (a) paths from b to e of length 3.
- (b) paths from a to c of length 5.
- (c) paths from d to b of length 6.



Solution:

Using alphabetical order to determine the position of the rows and columns, we have the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

To answer these questions we need to find A^3 , A^5 , and A^6 and then select the appropriate entries:

$$A^3 = \begin{pmatrix} 10 & 9 & 9 & 9 & 9 & 9 \\ 9 & 4 & 7 & 5 & 5 & 7 \\ 9 & 7 & 4 & 7 & 5 & 5 \\ 9 & 5 & 7 & 4 & 7 & 5 \\ 9 & 5 & 5 & 7 & 4 & 7 \\ 9 & 7 & 5 & 5 & 7 & 4 \end{pmatrix}, \quad A^5 = \begin{pmatrix} 140 & 101 & 101 & 101 & 101 & 101 \\ 101 & 64 & 72 & 67 & 67 & 72 \\ 101 & 72 & 64 & 72 & 67 & 67 \\ 101 & 67 & 72 & 64 & 72 & 67 \\ 101 & 67 & 67 & 72 & 64 & 72 \\ 101 & 72 & 67 & 67 & 72 & 64 \end{pmatrix},$$

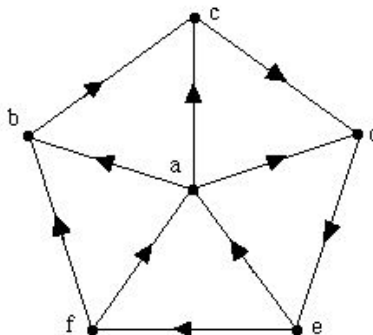
$$A^6 = \begin{pmatrix} 305 & 342 & 342 & 342 & 342 & 342 \\ 342 & 245 & 232 & 240 & 240 & 232 \\ 342 & 232 & 245 & 232 & 240 & 240 \\ 342 & 240 & 232 & 245 & 232 & 240 \\ 342 & 240 & 240 & 232 & 245 & 232 \\ 342 & 232 & 240 & 240 & 232 & 245 \end{pmatrix}.$$

- (a) The 2, 5-entry of A^3 is 5.
- (b) The 1, 3-entry of A^5 is 101.
- (c) The 4, 2-entry of A^6 is 240.

p.629, icon at Example 14

#2. Use powers of the adjacency matrix to find the following numbers of paths in this digraph:

- (a) paths from e to c of length 3.
- (b) paths from e to e of length 4.
- (c) paths from f to b of length 6.



Solution:

Using alphabetical order to determine the position of rows and columns, we have the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$A^3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 1 & 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}, \quad A^6 = \begin{pmatrix} 2 & 3 & 5 & 6 & 3 & 1 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 1 & 0 & 1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 2 & 3 & 2 \\ 5 & 5 & 5 & 4 & 2 & 3 \\ 3 & 5 & 5 & 4 & 1 & 1 \end{pmatrix}.$$

Selecting the appropriate entries, we have

- (a) the 5, 3-entry of A^3 is 3.
- (b) the 5, 5-entry of A^4 is 2.
- (c) the 6, 2-entry of A^6 is 5.

p.629, icon at Example 14

#3. Let G be an undirected simple graph with n vertices and adjacency matrix A . Suppose G is drawn with straight lines, none of which cross, such that no three vertices are collinear. Let $A^3 = [b_{i,j}]$. Explain why

$$\frac{b_{1,1} + b_{2,2} + \cdots + b_{n,n}}{6} = \text{the number of triangles in the figure.}$$

Solution:

Each triangle in the figure can be regarded as a circuit of length three. Also, each triangle can be described as six circuits of length three. For example, a triangle with vertices A, B, and C can be thought of as each of these circuits:

A-B-C-A, B-C-A-B, C-A-B-C, A-C-B-A, C-B-A-C, B-A-C-B.

Each entry $b_{i,i}$ in A^3 counts the number of circuits of length three that begin and end at vertex i . Therefore, adding the entries $b_{i,i}$ counts each triangle six times. Hence the result follows.
