

# ADD-ON 16A

## ARROW'S IMPOSSIBILITY THEOREM

What is the basis for using a particular social welfare function? One view is that a social welfare function represents subjective judgments about the appropriate distribution of consumption in a particular context. But some people aren't satisfied with that answer. They argue that concepts like equity and efficiency are universal. In their view, a social welfare function should reflect the consistent application of general principles, no matter what the context.

Rawlsians and utilitarians take this second view. For Rawlsians, the general principle is to give priority to the worst-off member of society; for utilitarians, it is to maximize total happiness. Both these principles assume the existence of meaningful cardinal measures of utility, an assumption that we've seen is hard to justify. What general principles can we apply when all our information about preferences is ordinal?

### THE THEOREM

In 1951, Kenneth Arrow (mentioned previously in Chapter 16) proved an astonishing mathematical result, known today as **Arrow's Impossibility Theorem**. Arrow set out to find reasonable principles that groups of two or more people could use when making choices involving at least three alternatives. He insisted that the group's decision should depend only on the ordinal preferences of each member, and not on cardinal measures of well-being, which he considered meaningless (as do most contemporary economists). He also insisted that the group should be able to order the alternatives from best to worst (with ties allowed), just as an individual can.<sup>1</sup> He then asked whether there are any reasonable procedures for converting the individuals' preference rankings into a single ranking for the group.

According to Arrow, any reasonable procedure should respect the following four principles:

1. *Nondictatorship*: The group's ranking shouldn't *always* be the same as the ranking of any particular individual.
2. *Pareto efficiency*: If everyone in the group ranks alternative *X* above alternative *Y*, then the group should rank *X* above *Y*.
3. *Independence of irrelevant alternatives*: The relative positions of any two alternatives in the group's ranking should depend only on the relative positions of those alternatives in each individual's ranking, and not on their positions relative to any other alternatives.

<sup>1</sup>In other words, he assumed that social comparisons between pairs of alternatives should be both complete and transitive. See Chapter 4, footnote 1, page 93.

According to **Arrow's Impossibility Theorem**, there is no reasonable procedure for converting the preference rankings of a group's members into a single ranking for the group.

4. *Unrestricted domain*: The procedure should apply regardless of how the group members rank the alternatives.

According to Arrow's thoroughly depressing theorem, there is no procedure for converting the individuals' preference rankings into a single ranking for the group that follows all four principles! Either we must give up at least one of these principles, or we must resign ourselves to evaluating social alternatives case-by-case.

## SOME EXAMPLES

There are, of course, many ways to make group decisions based on the preference rankings of the group members. However, every conceivable rule either fails to deliver a sensible ranking for the group, or violates at least one of Arrow's principles. Here are some examples.

### Majority Rule

Many people think that majority rule is a reasonable procedure for making group decisions. According to this procedure, the group should rank alternative  $X$  above alternative  $Y$  if a majority of its members rank  $X$  above  $Y$ . This simple and appealing rule runs afoul of Arrow's theorem because, in many situations, it fails to produce a sensible social ranking.

To illustrate, let's suppose that three individuals, Brad, Janet, and Rocky, have formed a carpool. While driving, they must listen to one of three types of music: rap, classical, or country western. Table 16A.1 lists their preference rankings. Notice that a majority (Brad and Rocky) prefer rap to classical; another majority (Brad and Janet) prefer classical to country western; and another majority (Janet and Rocky) prefer country western to rap. So, according to majority rule, the group ranks rap above classical, classical above country western, and country western above rap. Clearly, that is not a sensible ranking!

### The Borda Rule

Another seemingly reasonable procedure for making group decisions is known as the Borda rule.<sup>2</sup> This procedure assigns points to each alternative, and then ranks them according to their point totals. An alternative receives one additional point for each individual who ranks it first, two additional points for each individual who ranks it second, and so forth. A lower point total implies a higher position in the group's ranking. College football polls frequently use this procedure to rank teams.<sup>3</sup>

**Table 16A.1**  
**Music Preferences**

Alternative	Brad's Ranking	Janet's Ranking	Rocky's Ranking
Rap	1	3	2
Classical	2	1	3
Country Western	3	2	1

<sup>2</sup>The rule is named for the French scholar Jean-Charles Chevalier de Borda, who proposed it in 1770.

<sup>3</sup>College football polls typically assign higher points for higher ranks; a higher point total then implies a higher position in the poll's ranking. That procedure is equivalent to the one described in the text.

**Table 16A.2**  
Restricted Music Preferences

Alternative	Brad's Ranking	Janet's Ranking	Rocky's Ranking
Rap	1	2	1
Classical	2	1	2

The Borda rule runs afoul of Arrow's theorem because it doesn't respect one of the principles listed above: independence of irrelevant alternatives. To see why, let's reexamine the problem facing Brad, Janet, and Rocky. For the preference rankings listed in Table 16A.1, the Borda rule assigns  $1 + 2 + 3 = 6$  points to each alternative. In other words, it places them all in a tie. But what if the country western radio station were to go off the air? In that case, the group members would rank their remaining alternatives as shown in Table 16A.2. The Borda rule would then assign  $1 + 2 + 1 = 4$  points to rap, and  $2 + 1 + 2 = 5$  points to classical. Therefore, according to the Borda rule, the group should rank rap in a tie with classical when country western is available, but should rank rap above classical when country western is unavailable. Here, an "irrelevant alternative"—country western—affects the relative rankings of rap and classical music.

## POSSIBLE SOLUTIONS

Some people feel that Arrow's principles are more demanding than they should be, and that, as a result, they rule out perfectly reasonable procedures for converting the preference rankings of a group's members into a single group ranking. The two most controversial principles are independence of irrelevant alternatives and unrestricted domain.

### Gauging the Intensity of Preferences

Let's start with Arrow's third principle, independence of irrelevant alternatives. Suppose we present Brad, Janet, and Rocky with a fourth alternative: listen to an amplified 90-decibel recording of fingernails scratching on a blackboard—an experience which virtually everyone equates with torture. Table 16A.3 lists each individual's preference rankings over the expanded set. Notice that Brad thinks country western music is worse than fingernails on a blackboard; Janet feels the same way about rap. In comparing classical to rap, or classical to country western, comparisons of each alternative to fingernails on a blackboard are arguably relevant because they tell us something about the *intensity* of

**Table 16A.3**  
Expanded Music Preferences

Alternative	Brad's Ranking	Janet's Ranking	Rocky's Ranking
Rap	1	4	2
Classical	2	1	3
Country Western	4	2	1
Fingernails on blackboard	3	3	4

each individual's preferences. Based on the principle that no one's experience should be intolerable, the group might reasonably settle on classical music, the only alternative which everyone finds at least somewhat palatable (in the sense that they prefer it to fingernails on a blackboard).

Have we solved the conceptual problem raised by Arrow's theorem? Not necessarily. Comparisons with the fourth alternative are helpful only if we associate it with a level of well-being that is comparable across the group's members. How would we ever know if their experiences are actually comparable? If Brad is hard of hearing, he may not experience much discomfort when listening to fingernails on a blackboard. In contrast, if Rocky is hypersensitive to certain frequencies of sound, he may find the same experience absolutely excruciating. Because different people feel differently about listening to fingernails on a blackboard, the preference rankings shown in Table 16A.3 do not necessarily tell us whether Brad tolerates country western more or less easily than Rocky tolerates classical music.

### Restricting the Domain

Now let's turn to Arrow's fourth principle, unrestricted domain. In economics, we often know something about the nature of consumers' preferences—for example, that more is better (as discussed in Section 4.2), or that indifference curves have declining marginal rates of substitution (as discussed in Section 4.3). The principle of unrestricted domain nevertheless requires us to consider the possibility that a group member might have any *conceivable* preference ranking, even one that seems patently ridiculous. If we restrict our attention to *reasonable* preference rankings, we may be able to overcome the problems raised by Arrow's theorem, at least in some situations.

To illustrate this point, let's suppose that Brad, Janet, and Rocky have settled on classical music and must now choose a volume level. There are three alternatives: loud, soft, and medium. Do we need to consider every possible ranking over these alternatives? Arguably, anyone who prefers loud to medium will also prefer medium to soft, and anyone who prefers soft to medium will prefer medium to loud. In other words, it may be reasonable to assume that medium is no one's least preferred alternative. In that case, majority rule is a perfectly reasonable procedure for converting the preference rankings of the group's members into a group ranking.

Previously, we saw that majority rule may generate a nonsensical group ranking—one with a cycle. But if each individual's preferences satisfy the assumption listed in the last paragraph, cycles cannot arise. For example, if the group ranks medium above soft, and loud above medium, it cannot rank soft above loud. Why not? If a majority of the individuals prefer loud to medium, and if medium is no one's least preferred alternative, then each member of that same majority must rank loud as their top choice. Therefore, a majority of the members prefer loud to soft, which means the group ranks loud above soft, not below it. We will make the same point more generally in Chapter 20, where we discuss an important result known as the *median voter theorem*.