Mathematical Prerequisites

- Algebra (Appendix E, Mathematics Supplement)
- Geometry (Appendix I, Mathematics Supplement)

CHAPTER 4

Problem Solving

Engineers are problem solvers; employers hire them specifically for their problem-solving skills. As essential as problem solving is, it is impossible to teach a specific approach that will always lead to a solution. Although engineers use science to solve problems, this skill is more art than science. The only way to learn problem solving is to do it; thus, your engineering education will require that you solve literally thousands of homework problems.

In the modern world, computers are often used to aid problem solving. The novice student may think that the computer is actually solving the problem, but this is untrue. Only a human can solve problems; the computer is merely a tool.

4.1 TYPES OF PROBLEMS

A *problem* is a situation, faced by an individual or a group of individuals, for which there is no obvious solution. There are many types of problems we confront:

- *Research problems* require that a hypothesis be proved or disproved. A scientist may hypothesize that CFCs (chlorofluorocarbons) are destroying the earth's ozone layer. The problem is to design an experiment that proves or disproves the hypothesis. If you were confronted with this research problem, how would you approach it?
- *Knowledge problems* occur when a you encounter a situation that you do not understand. A chemical engineer may notice that the chemical plant produces more product when it rains. The cause is not immediately obvious, but further investigation might reveal that heat exchangers are cooled by the rain and hence have more capacity.
- *Troubleshooting problems* occur when equipment behaves in unexpected or improper ways. An electrical engineer may notice that an amplifier has a 60-cycle hum whenever the fluorescent lights are turned on. To solve this problem, she determines that extra shielding is required to isolate the electronics from the 60-cycle radiation emitted by the lights.
- *Mathematics problems* are frequently encountered by engineers, whose general approach is to describe physical phenomena with mathematical models. If a physical phenomenon can be described accurately by a mathematical model, the engineer unleashes the extraordinary power of mathematics, with its rigorously proved theorems and algorithms, to help solve the problem.

- *Resource problems* are always encountered in the real world. It seems there is never enough time, money, people, or equipment to accomplish the task. Engineers who can get the job done in spite of resource limitations are highly prized and well rewarded.
- *Social problems* can impact engineers in many ways. A factory may be located where there is a shortage of skilled labor because the local schools are of poor quality. In this environment, an engineer running a training program for factory workers must design the program to accommodate the low reading abilities of the trainees.
- *Design problems* are the heart of engineering. To solve them requires creativity, teamwork, and broad knowledge. A design problem must be properly posed. If your boss said, "Design a new car," you would not know whether to design an economy car, a luxury car, or a sport/utility vehicle. A well-posed design problem must include the ultimate objectives of the design project. If the boss said, "Design a car that goes from 0 to 60 miles per hour in 6.0 seconds, gets 50 miles-per-gallon fuel economy, costs less than $20,000, meets government pollution standards, and appeals to aesthetic tastes," then you could begin the project—even though it has difficult objectives.

4.2 PROBLEM-SOLVING APPROACH

The approach to solving an engineering problem should proceed in an orderly, stepwise fashion. The early steps are qualitative and general, whereas the later steps are more quantitative and specific. The elements of problem solving can be described as follows:

1. *Problem identification* is the first step toward solving a problem. For students, this step is done for them when the professor selects the homework problems. In the real engineering world, this step is often performed by a manager or creative engineer.

As an example, the management of an automotive firm may be painfully aware that the firm is losing market share. They challenge the engineering staff to design a revolutionary automobile to gain back lost sales.

2. *Synthesis* is a creative step in which parts are integrated together to form a whole.

For example, the engineers may determine that they can meet the design objectives for the new car (high fuel economy and rapid acceleration) by combining a highly efficient engine with a sleek, aerodynamic body.

3. *Analysis* is the step where the whole is dissected into pieces. Most of your formal engineering education will focus on this step. A key aspect of analysis is to translate the physical problem into a mathematical model. Analysis employs logic to distinguish truth from opinion, detect errors, make correct conclusions from evidence, select relevant information, identify gaps in information, and identify the relationship among parts.

For example, the engineers may compare the drag of a number of different body types and determine if the engine can fit under the hood of each body.

4. ***Application*** is a process whereby appropriate information is identified for the problem at hand.

For example, the engineers determine that a key question is to find the required force needed to propel the automobile at 60 mph at sea level, knowing the car has a projected frontal area of 19 ft^2 and a drag coefficient of 0.25.

5. ***Comprehension*** is the step in which the proper theory and data are used to actually solve the problem.

For example, the engineers determine that the drag force F on the automobile may be calculated using the formula

$$F = \tfrac{1}{2}C_d \rho A v^2$$

where C_d is the drag coefficient (dimensionless), ρ is the air density (kg/m^3), A is the projected frontal area (m^2), v is the automobile velocity (m/s), and F is the drag force (N). From the data, the force required to overcome air drag is

$$F = \tfrac{1}{2}(0.25)\left(1.18\,\frac{\text{kg}}{\text{m}^3}\right)\left[19\ \text{ft}^2 \times \left(\frac{\text{m}}{3.281\ \text{ft}}\right)^2\right]$$

$$\times\left(60\,\frac{\text{mi}}{\text{h}} \times \frac{\text{h}}{3600\ \text{s}} \times \frac{5280\ \text{ft}}{\text{mi}} \times \frac{\text{m}}{3.281\ \text{ft}}\right)^2 \times \frac{\text{N}}{\dfrac{\text{kg}\cdot\text{m}}{\text{s}^2}}$$

$$= 190\ \text{N} \times \frac{\text{lb}_\text{f}}{4.448\ \text{N}} = 42\ \text{lb}_\text{f}$$

The required force to overcome air drag is 190 newtons (in the metric system) or 42 pounds-force (in the American Engineering System). (*Note:* The above calculation involves many **conversion factors** from Appendix A. If you are uncomfortable with the conversions, do not worry; they will be discussed in more detail later in the book. You may wish to look at "A Word About Units" to review some basics regarding units.)

A Word about Units

Undoubtedly, you have been exposed to units of measure in high school. Here, our purpose is to refresh your memory. You are certainly familiar with the formula relating distance d, speed s, and time t,

$$d = st$$

Suppose your speed is 60 miles per hour and your trip takes 2 hours. We can calculate the distance traveled as

$$d = \frac{60\ \text{miles}}{\cancel{\text{hour}}} \times 2\ \cancel{\text{hour}} = 120\ \text{miles}$$

The hours cancel, leaving units of miles. Some students prefer to show their calculations as follows:

$$d = \frac{60\ \text{miles}}{\cancel{\text{hour}}}\ \Big|\ 2\ \cancel{\text{hour}} = 120\ \text{miles}$$

Either approach is correct as long as you are careful with your units.

If you wish to express the distance in kilometers, then a conversion factor is required. There is about 0.6 mile in a kilometer. Use this relationship to convert miles to kilometers:

$$d = 120\ \cancel{\text{miles}} \times \frac{\text{kilometer}}{0.6\ \cancel{\text{mile}}} = 200\ \text{kilometers}$$

When a unit is raised to a power, the conversion factor also must be raised to a power. For example, if a large land area were 120 square miles, then the area would be converted to square kilometers as follows:

$$A = 120\ \cancel{\text{miles}^2} \times \left(\frac{\text{kilometer}}{0.6\ \cancel{\text{mile}}}\right)^2 = 333\ \text{kilometers}^2$$

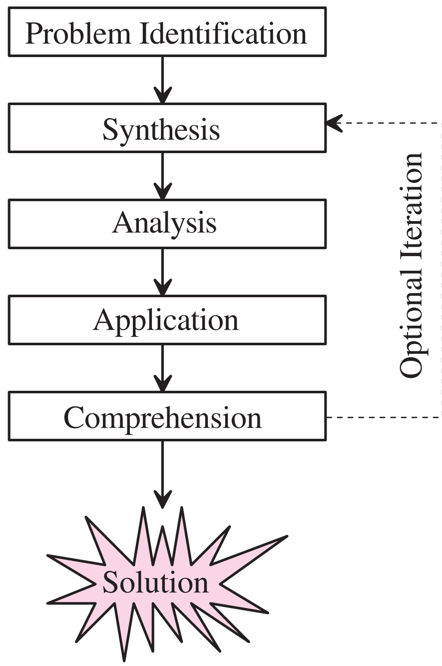


FIGURE 4.1
Problem-solving approach.

Although we would like to believe that these five steps can be followed in a linear sequence that always leads us to the correct solution, such is not always the case. Often, problem solving is an **iterative procedure,** meaning the sequence must be repeated because information learned at the end of the sequence influences decisions early in the sequence (Figure 4.1). For example, if the engineers determine that the force calculated in Step 5 is too high, they would have to return to earlier steps and try again.

4.3 PROBLEM-SOLVING SKILLS

Problem solving is a process in which an individual or a team applies knowledge, skills, and understanding to achieve a desired outcome in an unfamiliar situation. The solution is constrained by physical, legal, and economic laws as well as by public opinion.

To become a good problem solver, the engineer must have the following:

- Knowledge (first acquired in school, but later on the job).
- Experience to wisely apply knowledge.
- Learning skills to acquire new knowledge.
- Motivation to follow through on tough problems.
- Communication and leadership skills to coordinate activities within a team.

Table 4.1 compares skilled and novice problem solvers. Among the most important capabilities of a skilled problem solver is **reductionism,** the ability to logically break a problem into pieces. (Question: How do you eat an elephant? Answer: One bite at a time.)

TABLE 4.1
Comparison of skilled and novice problem solvers

Characteristic	Skilled Problem Solver	Novice Problem Solver
Approach	Motivated and persistent	Easily discouraged
	Logical	Not logical
	Confident	Lacks confidence
	Careful	Careless
Knowledge	Understands problem requirements	Does not understand problem requirements
	Rereads problem	Relies on a single reading
	Understands facts and principles	Cannot identify facts and principles
Attack	Breaks the problem into pieces*	Attacks the problem all at once
	Understands the problem before starting	Tries to calculate the answer right away
Logic	Uses basic principles	Uses intuition and guesses
	Works logically from step to step	Jumps around randomly
Analysis	Organized	Disorganized
	Thinks carefully and thoroughly	Hopes the answer will come
	Clearly defines terms	Uncertain about the meaning of symbols
	Careful about relationships and meaning of terms	Jumps to unfounded conclusions about the meanings of terms
Perspective	Has a feel for the correct magnitude of answers	Uncritically believes the answers produced by the calculator or computer
	Understands the differences between important and unimportant issues	Cannot differentiate between important and unimportant issues
	Uses rule of thumb to estimate the answer	Cannot estimate the answer

* Very important
Adapted from: H. S. Fogler, "The Design of a Course in Problem-Solving," in *Problem Solving,* AIChE Symposium Series, vol. 79, no. 228, 1983.

Reductionism contrasts with *synthesis,* the creative process of putting pieces together. If your problem were to design an airplane, you would use reductionism to design subsystems (engines, landing gear, electronic controls, etc.) and synthesis to combine the pieces together.

4.4 TECHNIQUES FOR ERROR-FREE PROBLEM SOLVING

All students hope to do error-free problem solving, because it assures them of excellent grades on homework and exams. Further, error-free calculations will be required when the student enters the engineering workforce. The truth is that engineers can never be certain that their answers are correct. A civil engineer who goes through elaborate calculations to design a bridge cannot be certain of her calculations until a heavy load is placed on the bridge and the deflection agrees with her calculations. Even then there is uncertainty, because there may be errors in the calculations that tend to cancel each other out.

Although we can never be certain our answer is correct, we can increase the probability of calculating a correct answer using the following procedure:

Given:

1. Always draw a picture of the physical situation.
2. State any assumptions.
3. Indicate all given properties on the diagram ***with their units.***

Find:

4. Label unknown quantities with a question mark.

Relationships:

5. From the text, write the *main equation* that contains the desired quantity. (If necessary, you might have to derive the appropriate equation.)
6. Algebraically manipulate the equation to isolate the desired quantity.
7. Write *subordinate equations* for the unknown quantities in the main equation. Indent to indicate that the equation is subordinate. You may need to go through several levels of subordinate equations before all the quantities in the main equation are known.

Solution:

8. After all algebraic manipulations and substitutions are made, insert numerical values ***with their units***.
9. Ensure that units cancel appropriately. Check one last time for a sign error.
10. Compute the answer.
11. Clearly mark the final answer. ***Indicate units.***
12. Check that the final answer makes physical sense!
13. Ensure that all questions have been answered.

Notice that the first step in solving an engineering problem is to draw a picture. We cannot overstate the importance of graphics in engineering. Solving most engineering problems requires good visualization skills. Further, communicating your solution requires the ability to draw.

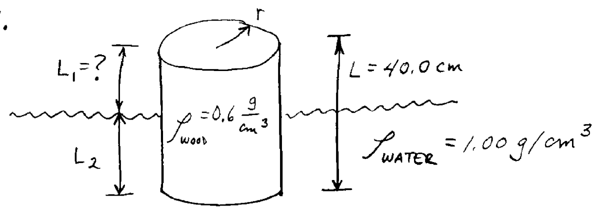
The above procedure is a process of working backward. You start from the end with the unknown, desired quantity and work backward using the given information. Not every problem fits into the paradigm described above, but if you use it as a guide, it will increase the likelihood of obtaining a correct answer. Notice that units are emphasized. Most calculation errors result from a mistake with units.

It goes without saying that the solution should be as neat as possible so it can be easily checked by another person (a co-worker or boss). Calculations should be performed in pencil so changes can be made easily. It is recommended that you use a mechanical pencil because it does not need to be sharpened. A fine (0.5-mm diameter), medium-hardness lead works best. Any erasures need to be complete.

Most engineering professors prefer that students use **engineering analysis paper,** because it has a light grid pattern that can be used for graphing or drafting. It also has headings and margins that allow the calculations to be labeled and documented. Use one side of the paper only. If at all possible, complete the solution on a single page; this allows for easy checking. Sample Problems 1, 2, and 3 present examples of well-solved homework problems.

	3 OCT 2007	HOMEWORK # 3	JOE ENGINEER 000-00-0000 ENGR 109 SECTION 521	1 OF 3

1.



$m_{WOOD} = m_{WATER}$ (ARCHIMEDES PRINCIPLE)

$$m_{WOOD} = \rho_{WOOD} V_{WOOD}$$

$$V_{WOOD} = \pi r^2 L$$

$$m_{WOOD} = \rho_{WOOD} \pi r^2 L$$

$$m_{WATER} = \rho_{WATER} V_{WATER}$$

$$V_{WATER} = \pi r^2 L_2$$

$$L = L_1 + L_2$$

$$L_2 = L - L_1$$

$$V_{WATER} = \pi r^2 (L - L_1)$$

$$m_{WATER} = \rho_{WATER} \pi r^2 (L - L_1)$$

$$\rho_{WOOD} \cancel{\pi r^2} L = \rho_{WATER} \cancel{\pi r^2} (L - L_1)$$

$$\rho_{WOOD} L = \rho_{WATER} (L - L_1) = \rho_{WATER} L - \rho_{WATER} L_1$$

$$\rho_{WATER} L_1 = \rho_{WATER} L - \rho_{WOOD} L = L(\rho_{WATER} - \rho_{WOOD})$$

$$L_1 = L \frac{\rho_{WATER} - \rho_{WOOD}}{\rho_{WATER}} = 40.0\ cm\ \frac{1.00\ g/cm^3 - 0.600\ g/cm^3}{1.00\ g/cm^3}$$

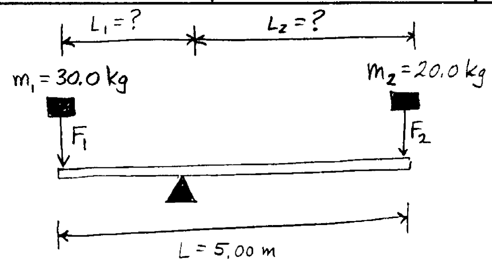
$$\boxed{L_1 = 16.0\ cm}$$

Sample Problem 1:

Archimedes' principle states that the total mass of a floating object equals the mass of the fluid displaced by the object. A 40.0-cm log is floating vertically in the water. Determine the length of the log that extends above the water line. The water density is 1.00 g/cm^3 and the wood density is 0.600 g/cm^3.

	3 OCT 2007	HOMEWORK #3	JOE ENGINEER 000-00-0000 ENGR 109 SECTION 521	2 OF 3

2.



ASSUME MASS OF TEETER-TOTTER IS NEGLIGIBLE

$F_1 L_1 = F_2 L_2$ @ STATIC EQUILIBRIUM

$$F_1 = m_1 g$$
$$F_2 = m_2 g$$

$$m_1 \not{g} L_1 = m_2 \not{g} L_2$$

$$L_1 = \frac{m_2}{m_1} L_2$$

$$L = L_1 + L_2$$
$$= \frac{m_2}{m_1} L_2 + L_2$$
$$= \left(\frac{m_2}{m_1} + 1\right) L_2$$

$$L_2 = \frac{L}{\frac{m_2}{m_1} + 1} = \frac{5.00\ \text{m}}{\frac{20.0\ \text{kg}}{30.0\ \text{kg}} + 1} = \boxed{3.00\ \text{m}}$$

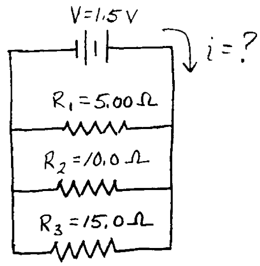
$$L_1 = L - L_2 = 5.00\ \text{m} - 3.00\ \text{m} = \boxed{2.00\ \text{m}}$$

Sample Problem 2:

An object is in static equilibrium when all the moments balance. (A moment is a force exerted at a distance from a fulcrum point.) A 30.0-kg child and a 20.0-kg child sit on a 5.00-m long teeter-totter. Where should the fulcrum be placed so the two children balance?

	3 OCT 2007	HOMEWORK # 3	JOE ENGINEER 000-00-0000 ENGR 109 SECTION 521	3 OF 3

3.



$$V = iR$$

$$i = \frac{V}{R}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$R = \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right]^{-1}$$

$$= \left[\frac{1}{5.00\,\Omega} + \frac{1}{10.0\,\Omega} + \frac{1}{15.0\,\Omega}\right]^{-1}$$

$$= 2.73\,\Omega$$

$$i = \frac{1.5\,V}{2.73\,\Omega} \times \frac{\Omega}{V/A}$$

$$\boxed{i = 0.55\,A}$$

Sample Problem 3:

The voltage drop through a circuit is equal to the current times the total resistance of the circuit. When resistances are placed in parallel, the inverse of each resistance sums to the inverse of the total resistance. Three resistors (5.00 Ω, 10.0 Ω, and 15.0 Ω) are placed in parallel. How much current flows from a 1.5-V battery?

4.5 ESTIMATING

The final step in solving a problem is to check the answer. One can accomplish this by working the problem using a completely different method, but often there is not time for this. Instead, a valuable approach is to estimate the answer.

The ability to estimate comes with experience. After many years of working similar problems, an engineer can "feel" if the answer is in the right ballpark. As a student, you do not have experience, so estimating may be difficult for you; however, your inexperience is not an excuse for failing to learn how. If you cultivate your estimating abilities, they will serve you well in your engineering career. It has been our experience that much business is conducted over lunch, using napkins for calculations and drawings. An engineer who has the ability to estimate will impress both clients and bosses.

A question we often hear from students is, How do I make reasonable assumptions during the estimation process? There is no quick and simple answer. One of the reasons that "has broad interests" and "collects obscure information" are included among the traits of a creative engineer (Chapter 1) is that these traits will supply you with raw data and cross-checks for reasonableness when making estimations. With practice and experience you will become better. Notice how much your skills improve with the few examples you do in the classroom and for homework. After working in a particular engineering field for a while, many estimations will become second nature.

To get you started as an expert estimator, we have prepared Table 4.2. It lists relationships we find to be very useful during our business lunches. Use this table as a starting point; as you mature in your own engineering discipline, you will undoubtedly learn other relationships and rules of thumb that will serve you well.

The following examples describe five approaches to estimation.

EXAMPLE 4.1 Simplify the Geometry

Problem Statement: Estimate the surface area of an average-sized man.

Solution: Approximate the human as spheres and cylinders.

$$A = A_{\text{head}} + A_{\text{torso}} + 2A_{\text{leg}} + 2A_{\text{arm}}$$

$$= 4\pi r_{\text{head}}^2 + (2\pi rL)_{\text{torso}} + 2(2\pi rL)_{\text{leg}} + 2(2\pi rL)_{\text{arm}}$$

$$= 4\pi(3.75 \text{ in})^2 + [2\pi(5 \text{ in})25 \text{ in}] + 2[2\pi(3 \text{ in})32 \text{ in}] + 2[2\pi(1.5 \text{ in})24 \text{ in}]$$

$$= 2620 \text{ in}^2 \times \left(\frac{2.54 \text{ cm}}{\text{in}}\right)^2 \times \left(\frac{\text{m}}{100 \text{ cm}}\right)^2 = 1.69 \text{ m}^2$$

This is very close to the accepted value of 1.7 m^2.

TABLE 4.2
Useful relationships for estimation purposes

Conversion Factors

1 ft = 12 in	1 atm = 760 mm Hg
1 in = 2.54 cm	1 atm ≈ 34 ft H_2O
1 mi = 5280 ft	1 min = 60 s
1 km ≈ 0.6 mi	1 h = 60 min
1 ft^3 = 7.48 gal	1 d = 24 h
1 gal = 3.78 L	1 year = 365 $\frac{1}{4}$ d
1 bbl = 42 gal	1 Btu ≈ 1000 J
1 mi^2 = 640 acre	1 Btu ≅ $\frac{1}{4}$ kcal
1 kg ≈ 2.2 lb_m	1 hp = 550 ft·lb_f/s
1 atm = 14.7 psi	1 hp ≈ 0.75 kW
1 atm ≈ 1×10^5 N/m^2	

Temperature Conversions

[K] = [°C] + 273.15	[°R] = [°F] + 459.67
[°F] = 1.8 [°C] + 32	[°C] = ([°F] − 32)/1.8

Ideal Gas

Standard temperature = 0°C
Standard pressure = 1 atm
Molar volume (@ STP) = 22.4 L/gmole
Molar volume (@ STP) = 359 ft^3/lbmole

Molecular Weights

H = 1	N = 14
C = 12	O = 16

Air = 29 (21 mol % O_2, 79 mol % N_2)

Geometry Formulas

Circle area = πr^2
Circle circumference = $2\pi r$
Cylinder volume = $\pi r^2 L$
Cylinder area (without end caps) = $2\pi rL$
Sphere area = $4\pi r^2$
Sphere volume = $\frac{4}{3}\pi r^3$
Triangle area = $\frac{1}{2}$ (base)(height)

Temperature References

Absolute zero = 0 K
He boiling point = 4 K
N_2 boiling point = 77 K
CO_2 sublimation point = 195 K = −78°C
Mercury melting point = −39°C
Freezer temperature = −20°C
H_2O freezing point = 0°C
Refrigerator temperature = 4°C
Room temperature = 20°C to 25°C
Body temperature = 37°C
H_2O boiling point = 100°C
Lead melting point = 327°C
Aluminum melting point = 660°C
Iron melting point = 1540°C
Flame temperature (air oxidant) = 2100°C
Flame temperature (pure oxygen) = 3300°C
Carbon melting point = 3700°C

Physical Constants

Speed of light ≈ 3×10^8 m/s
Speed of sound (air, 20°C, 1 atm) ≈ 770 mph
Avogadro's number ≈ 6×10^{23}
Acceleration from gravity ≈ 9.8 m/s^2 ≈ 32.2 ft/s^2

Mathematical Constants

$e \approx 2.718$	$\pi \approx 3.14159$

Water Properties

Density = 1 g/cm^3 = 62.4 lb_m/ft^3 = 8.34 lb_m/gal
Latent heat of vaporization ≈ 1000 Btu/lb_m
Latent heat of fusion ≈ 140 Btu/lb_m
Heat capacity = 1 Btu/(lb_m · °F) = 1 kcal/(kg · °C)
Melting point = 0°C
Boiling point = 100°C

Densities

Air (0°C, 1 atm) = 1.3 g/L	Aluminum = 2.6 g/cm^3
Wood ≈ 0.5 g/cm^3	Steel = 7.9 g/cm^3
Gasoline = 0.67 g/cm^3	Lead = 11.3 g/cm^3
Oil = 0.88 g/cm^3	Mercury = 13.6 g/cm^3
Concrete = 2.3 g/cm^3	

Earth

Circumference ≈ 40,000 km
Mass ≈ 6×10^{24} kg

People

Male	Female
Avg mass = 191 lb_m	Avg mass = 166 lb_m
Avg height = 69 in	Avg height = 64 in

Sustained work output of man = 0.1 hp

Energy

Natural gas: 1000 standard ft^3 ≈ 10^6 Btu
Crude oil: 1 bbl ≈ 6×10^6 Btu
Gasoline: 1 gal ≈ 125,000 Btu
Coal: 1 lb_m ≈ 10,000 Btu

Ultimate Strength of Materials

Steel = 60,000 to 125,000 lb_f/in^2
Aluminum = 11,000 to 80,000 lb_f/in^2
Concrete = 2000 to 5000 lb_f/in^2

Automobiles

Typical mass = 2000 to 4000 lb_m
Typical fuel usage = 600 gal/year
Typical fuel economy = 20 to 30 mi/gal
Typical power = 50 to 500 hp
Engine speed = 1000 to 5000 rpm

Typical Appliance Energy Usage

Fluorescent lamp = 40 W
Incandescent lamp = 50 to 100 W
Refrigerator-freezer = 500 W
Window air conditioner = 1500 W
Heat pump = 12,000 W
Cooking range = 12,000 W
Avg continuous home electricity usage ≈ 1 kW

EXAMPLE 4.2 Use Analogies

Problem Statement: Estimate the volume of an average-sized man.

Solution: Assume the density of a man is 0.95 that of water. (People are mostly water, but they do float slightly when swimming, so the density must be slightly less than water.)

$$V = \frac{m}{\rho} = \frac{191\ \mathrm{lb_m}}{0.95(1.0\ \frac{\mathrm{g}}{\mathrm{cm^3}})} \times \frac{\mathrm{kg}}{2.2\ \mathrm{lb_m}} \times \frac{1000\ \mathrm{g}}{\mathrm{kg}} \times \left(\frac{\mathrm{m}}{100\ \mathrm{cm}}\right)^3 = 0.091\ \mathrm{m^3}$$

EXAMPLE 4.3 Scale Up from One to Many

Problem Statement: How many bed pillows can fit in the back of a tractor trailer?

Solution: One pillow measures 3 in thick by 16 in wide by 21 in long. The cargo bed of a tractor trailer measures roughly 8 ft wide by 10 ft tall by 35 ft long.

$$\text{Number of pillows} = \frac{V_{\text{truck}}}{V_{\text{pillow}}} = \frac{(8\ \mathrm{ft})(10\ \mathrm{ft})(35\ \mathrm{ft})}{(3\ \mathrm{in})(16\ \mathrm{in})(21\ \mathrm{in})} \times \left(\frac{12\ \mathrm{in}}{\mathrm{ft}}\right)^3 = 4800$$

This calculation neglects the compression of the pillows as they stack on top of each other (which would increase the number carried). It also neglects the volume occupied by packing materials (which would decrease the number carried). One can hope that the errors due to the simplifying assumptions will tend to cancel each other out.

EXAMPLE 4.4 Place Limits on Answers

Problem Statement: Estimate the mass of an empty tractor trailer.

Solution: It seems reasonable that the tractor trailer mass should be more than five automobiles, but less than 30 automobiles.

$$\text{Lower bound} = 5 \times 3000\ \mathrm{lb_m} = 15{,}000\ \mathrm{lb_m}$$

$$\text{Upper bound} = 30 \times 3000\ \mathrm{lb_m} = 90{,}000\ \mathrm{lb_m}$$

Because automobiles typically have a mass between 2000 and 4000 $\mathrm{lb_m}$, an average value of 3000 $\mathrm{lb_m}$ was used. The estimated mass of the tractor trailer is reasonable; an Internet search revealed that heavy trucks weigh about 42 tons (84,000 $\mathrm{lb_m}$).

EXAMPLE 4.5 Extrapolate from Samples

Problem Statement: How much fuel is burned by Texas A&M students for all the Thanksgiving visits home?

Solution: A survey of 20 random students reveals that 6 come from Houston (100 miles away), 4 from Dallas (200 miles), 3 from Austin (90 miles), 2 from San Antonio (190 miles), and 5 live too far to return home for Thanksgiving. Assume that the average car has 1.5 occupants (half have two occupants and half have one occupant). The student population is 42,000 students.

$$\text{Fuel used} = \frac{\text{Total miles}}{\text{Average fuel economy}}$$

$$\text{Total miles} = \text{Houston} + \text{Dallas} + \text{Austin} + \text{San Antonio}$$

$$\text{Houston} = 2 \times \frac{1}{1.5} \times \frac{6}{20} \times 42{,}000 \times 100 \text{ mi} = 1{,}680{,}000 \text{ mi}$$

$$\text{Dallas} = 2 \times \frac{1}{1.5} \times \frac{4}{20} \times 42{,}000 \times 200 \text{ mi} = 2{,}240{,}000 \text{ mi}$$

$$\text{Austin} = 2 \times \frac{1}{1.5} \times \frac{3}{20} \times 42{,}000 \times 90 \text{ mi} = 756{,}000 \text{ mi}$$

$$\text{San Antonio} = 2 \times \frac{1}{1.5} \times \frac{2}{20} \times 42{,}000 \times 190 \text{ mi} = 1{,}064{,}000 \text{ mi}$$

$$\text{Total miles} = 1{,}680{,}000 \text{ mi} + 2{,}240{,}000 \text{ mi} + 756{,}000 \text{ mi} + 1{,}064{,}000 \text{ mi}$$

$$= 5{,}740{,}000 \text{ mi}$$

$$\text{Fuel used} = \frac{5{,}740{,}000 \text{ mi}}{25 \text{ mi/gal}} = 229{,}600 \text{ gal} \approx 230{,}000 \text{ gal}$$

4.6 CREATIVE PROBLEM SOLVING

The Accreditation Board for Engineering and Technology (ABET), in its *1985 Annual Report* (New York, ABET, 1986), defines engineering as "the profession in which a knowledge of the mathematical and natural sciences gained by study, experience, and practice is applied with judgment to develop ways to utilize, economically, the materials and forces of nature for the benefit of mankind." This rather painfully contrived definition (obviously the work of a committee) does, in fact, contain the necessary ingredients to define engineering: skill in mathematics and the natural sciences coupled with a knack for producing useful works with this knowledge. What is missing from the ABET definition is the necessity for creativity and problem-solving skills.

The scientist studies what nature has already created; the engineer creates from nature what did not exist before. In this creative process, the engineer marshals her skill in mathematics, her knowledge of materials, and the particular principles of her specialized engineering discipline, and from this resource creates a new solution for a human need or problem. In almost all cases, the solution is also constrained by economic reality: not only must it answer the particular need that called for a solution, but it must answer the need cheaply.

Somehow, engineering is often stereotyped as dull and stifling work. We are always astounded at how an occupation that builds machines to deliver people to the moon, that has provided means to solve most of the world's problems in transportation, communication, and basic life support, could be cast as dull.

Because creativity figures so centrally in good engineering, it is worthwhile to take another look at the nature of creativity, surely the most misunderstood process of the human intellect. Though possessed by all, many assume that "real" creativity is bestowed upon the few, principally writers and artists. Furthermore, creative expression is perceived to occur in a blinding flash, the creator bringing to life his vision by merely following that which he saw in the creative flash.

Neither of these ideas about creativity is true. Certainly it is true that a new insight will occur in an instant, and a whole new way of looking at things will present itself to the thinker; but this is not the first step or the last in the creative process and often not even the most important step. The greatest artist must spend years perfecting his mastery of the materials used in his field before his flashes of insight can be expressed as works of art. Then the expression of the original thought will often lead to other insights or, at a minimum, numerous creative choices on how to best express the original insight within the framework of skills and materials at hand. Likewise, in engineering, the beginner must master the basics and "practice" engineering before his insights can be brought to life.

This idea brings us back to the second misconception about creativity: that it is the domain of a few artists chosen by a higher power or good fortune. Nothing could be further from the truth. There are two types of problems in the world: those that must be solved all at once and those that can be broken into smaller parts and solved a piece at a time. People who solve the first type of problems are world-class geniuses, and each generation produces one or two such people. Fortunately, most problems are of the second class and lie within the range of the rest of us.

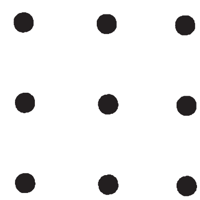
Also, creativity is not a process in which the solution to the problem leaps into existence, but a process in which a myriad of small parts of the whole are solved, until the end product stands complete. This skill at managing small pieces of a larger problem is called by many names; for our discussion, we use the term **problem solving.** Problem-solving skills arise from innate talent, from training, and, most importantly, from long practice. On the whole, persistence and practice are far greater gifts than superior intellect. The world is full of intellectually brilliant failures. On the other hand, few fail who combine perseverance with a bit of good judgment.

This section presents discussions of creativity, followed by a selection of general problems to allow development and practice of problem-solving skills (because we believe that these skills can be taught) with the hope that these general methods will be applied in your later efforts in engineering. Realize that the words *problem solving* have been applied to a long list of activities ranging from simple arithmetic problems to complex open-ended engineering applications. For now we are using the term to mean an activity or thought process that brings to a satisfying conclusion the problems presented.

4.6.1 Creative Leaps, Aha! Experiences, and Sideways Thinking

One of our favorite puzzles was first constructed by Sam Lloyd (1841–1911), who was probably the most prolific puzzlist in America. Sam presented this puzzle to a client who wanted to use it for advertising, and Sam offered to give it to the advertiser for free if he could solve it overnight. The next day Sam received $1000 and showed the advertiser the solution. Here's the puzzle; see if you can save yourself $1000:

Connect the following nine dots with four contiguous straight lines. Draw the four straight lines without lifting your pencil.



Our chief reason for admiring the puzzle is that its solution is without tricks and completely simple and straightforward; but in trying to work it out we block ourselves from seeing the answer. (We've even had a "proof" brought to us that it cannot be done.) During lecture we often use this puzzle to illustrate how we draw rings around ourselves and prevent bringing our full faculties to focus on the problem. With these hints, see if you can do it.

If you solved the puzzle, perhaps you sensed the thrill of accomplishment that we feel when we finally have the necessary insight to see a solution to a problem we've wrestled with. The world we live in has so many first-class problems to tackle, and, for many engineers, a large part of the motivation in life arises from the feelings we experience when solving a particularly knotty problem. If you enjoy solving puzzles, you'll enjoy engineering.

Many books have been written on creative thought, ranging in quality from psychobabble to serious discussions of the nature of creativity. In this text, we can only expose you to a few thoughts on creative endeavors. We will concentrate on an area of creativity called by many names: *Aha! thinking, lateral* or *sideways thinking,* or *flash thinking.* Basically all of these names describe a type of thinking that has little or nothing to do with the linear, logical processes of reasoning. With this type of thinking, the perception of the world is suddenly given a half-twist, and the thinker experiences an insight that allows a solution to the problem at hand.

Following is a series of problems for you to solve, both to give you practice in creative efforts and to analyze and improve the methods you use to creatively solve problems. These first problems are of a class we call **manipulative models.** These problems are similar to working a jigsaw puzzle: basically, you need some physical entity to manipulate in order to solve the problem. Crick and Watson, when they were working out the complicated structure of DNA, had a Tinkertoy model on which to test out their various theories (the model is now in the British Museum of Science). When complicated layouts are to be built (for the control panel for a nuclear reactor or the flight deck of the space shuttle), mockups are made first so that problems can be solved on the model.

So here are your problems:

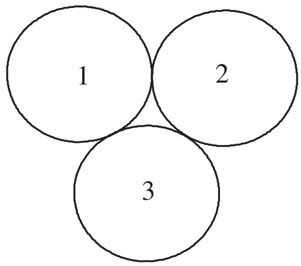
PROBLEM 4.1

Using six short cylinders (nickels or quarters will do as models) make each touch:

Two and only two others (two completely different ways)
Three and only three others (two completely different ways)
Four and only four others (two completely different ways)

"Touch" is contact between two cylinders either along the curved perimeter or along the flat faces. Here is an example using three nickels and having each touch two and only two others:

1 touches 2 and 3;
2 touches 1 and 3;
3 touches 1 and 2.



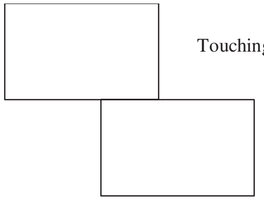
This is also the format for answering the problems: provide a sketch of the solution and a list of each object the other touches. "Completely different ways" means a new physical arrangement that cannot be reached by merely renumbering or rotating an old solution.

PROBLEM 4.2

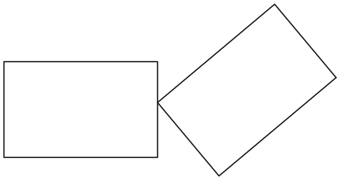
Continue with a similar exercise, but now use six rectangular **parallelepipeds** (matchboxes or books are appropriate models) and have each touch:

Two others (two completely different ways)
Three others (two completely different ways)
Four others
Five others

The "touch" definition differs for the parallelepipeds. To be touching, a flat surface must be in contact with another flat surface. Contact between corners or between a corner and a flat surface does not count as touching. For example:



Touching



Not Touching

PROBLEM 4.3

Finally, using six equilateral triangles with finite thickness (plane or paper-thin triangles are not the correct shape), make each touch:

Two others (two completely different ways)
Three others (two completely different ways)

As with rectangular parallelepipeds, corner touches do not count.

4.6.2 Classifying Problem-Solving Strategies

A good engineer's time is always spoken for. Because good engineers enjoy engineering, it is indeed an act of love of the profession to take time from their work to write books and articles for the general public. It is even rarer for engineers to analyze how they approach engineering problems and write this down for the student. In 1945, George Polya, a mathematician, published the book *How to Solve It* (Polya, 1945), describing in general terms methods to solve mathematical problems. These methods are well suited to engineering problems. (The Princeton Science Library reissued Polya's book in 2004. It is well worth reading.)

Polya summarized his method as the "How to Solve It List":

First. You have to *understand* the problem.

What is the unknown? What are the data? What is the condition?

Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

Second. Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should eventually obtain a *plan* of the solution.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use its results? Could you use its method? Should you introduce some auxiliary element in order to make its use possible? Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, and drop the other part; how far is the unknown then determined, and how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the

TABLE 4.3
Problem-solving strategies

Polya	Woods et al.	Bransford and Stein	Schoenfeld	Krulik and Rudnick
Understand the problem	Define the problem	Identify the problem	Analyze the problem	Read the problem
	Think about it	Define and represent it	Explore it	Explore it
Devise a plan	Plan	Explore possible strategies	Plan	Select a strategy
Carry out the plan	Carry out the plan	Act on the strategies	Implement	Solve
Look back	Look back	Look back and evaluate the effects of your activities	Verify	Look back

unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Third. *Carry out* your plan.

Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

Fourth. *Examine* the solution obtained.

Can you *check the result?* Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result, or the method, for some other problem?

There are many variations of this basic scheme; Table 4.3 summarizes strategies suggested by other authors whose work you will find in the bibliography. Most notable is that they all offer roughly the same advice, and that all of them emphasize *checking your results* as the last step. To err is human, but most human of all is to screw everything up in the final steps after the fun of solving the problem is gone and deadlines gallop closer. Many an engineer wishes he had taken an extra 5 minutes to check units or to see if the result really made sense.

Many problem-solving methods have been categorized. The following examples are by no means a complete list—but the particular problem one is pondering has a vanishingly small chance of being categorized anyway. The object of these examples is to get you to think about general methods of attack on problems and, let us hope, to ponder your own methods of tackling problems. Moving from abstract discussions of problem solving to concrete examples will help you to grasp each strategy.

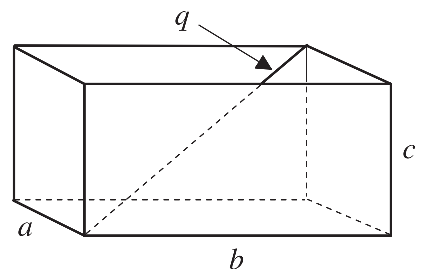
EXAMPLE 4.6 Exploit Analogies or Explore Related Problems

Exploiting analogies is one of the most common approaches to solving a problem. In the old days of engineering (30 years ago), machines called analog computers existed. By using resistors to model fluid friction, capacitors to model holdup tanks, and batteries for pumps, the flow of fluid through a piping system could be modeled with electronics. As you practice engineering, your list of solved problems will grow and become your list against which to test new problems, looking for similarities and analogies.

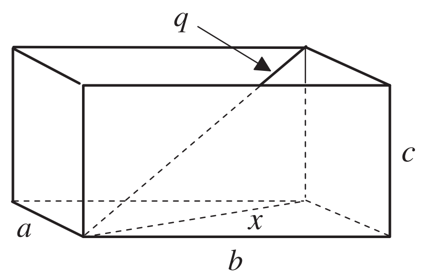
Problem Statement: Freight carriers charge by rate proportional to the length, depth, and height of a package. What are the minimum lengths, depths, and heights of a rectangular parallelepiped to ship a rod of length q, if the diameter of the rod is negligible? (Adapted from Polya, 1945, and others.)

Solution:

What is the unknown? The lengths of the sides of the parallelepiped, say a, b, and c.
What is the given? The length of the rod q, and that the diameter of the rod is negligible.
Draw a figure.



What is the unknown? The diagonal of the parallelepiped.
Do you know any problems with a similar unknown? Yes, sides of right triangles!
Devise a plan.



Carry out the plan. If we could find x, then we could relate c and x to q by the **Pythagorean theorem,** that is,

$$q^2 = x^2 + c^2$$

But x^2 is equal to $b^2 + a^2$, so that

$$q = \sqrt{a^2 + b^2 + c^2}$$

Look back. Did we use the appropriate data? Does the answer make sense? (What happens if a goes to zero? Check special cases.) Is there symmetry? (Because a, b, and c can be arbitrarily rotated to any edge, is the form of the answer symmetrical?)

EXAMPLE 4.7 Introduce Auxiliary Elements; Work Auxiliary Problem

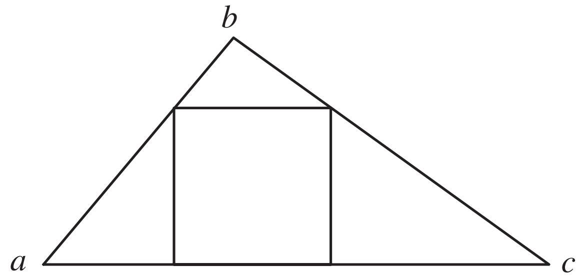
Sometimes a problem is too difficult (or it's too early in the morning) to tackle head on. An approach that allows progress and often lights the path to the real insight to unknot the problem is to tackle the problem with one or more of the constraints lifted. By solving the easier problem, we can sometimes maneuver the parameters of the simpler problem until the lifted constraint is also satisfied and we're done. As in this problem, it's spectacular when it works.

Problem Statement: Inscribe a square in any given triangle. Two vertices of the square should be on the base of the triangle, the other two vertices of the square on the other two sides of the triangle, one on each. (From Polya, 1945.)

Solution:

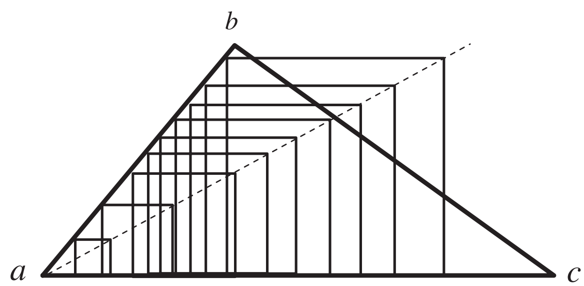
What is the unknown? A method to inscribe a square in any triangle.
What is the given? An arbitrary triangle. Directions for placing square in triangle.

Draw a figure.



Ouch! This is tough. Even using a graphics program to do the hard part, it still takes us 10 minutes to jockey a square about the right size into a triangle, and then it's only an approximation. With odd-shaped triangles (short height with a long base, or tall with a narrow base), the whole thing gets worse.

Hmmm . . . What if we let squares sit on the base, but they only have to touch the triangle on side *ab*? If we draw a bunch of squares, maybe the lights will come on.



Notice anything unusual about the free vertices? They all lie along a single line! From there, algorithm construction is simple.

Carry out the plan. Construct an arbitrary square on the base of the triangle and draw a line from point *a* on the triangle through the free vertex of the square. Where this line intersects side *bc* on the triangle is where the free vertex of the desired square will coincide with side *bc* of the triangle. Construct a perpendicular line to the base of the triangle through this point. This is one side of the square. Mark off the same distance along the base of the triangle; this is the second side of the square. Construct a perpendicular from this point on the base to side *ab* on the triangle. This locates all four vertices of the square.

Look back. With the clarity of hindsight, why do the vertices lie on a single line? Is this right? Knowing this, can an easier algorithm be developed rather than constructing an arbitrary square on the base of the triangle and drawing a line from point *a* on the triangle through the free vertex of the square?

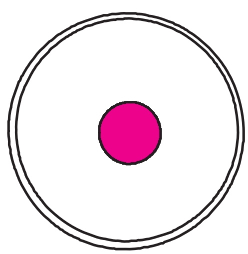
EXAMPLE 4.8 Generalizing: The Inventor's Paradox

Sometimes it is easier to solve a more general version of a given engineering problem and put in the specific parameters at the end, rather than to solve the particular problem straight away. Much work has been done to solve general problems, so the efforts of others can sometimes be brought to your aid by enlarging your problem. In the study of differential equations, for example, the trick is to identify which general class of equations the particular equation you're working with belongs. Once you've done this, the answer is obvious, because general solutions are given for each class of equations.

Problem Statement: A satellite in circular orbit about the earth, 8347.26 miles from the center of the earth, moves outward 6.1 ft. How much new area is enclosed in its new orbit?

Solution:

What is the unknown? The extra area inside the orbit.
What is the given? Circular orbit with a radius of 8347.26 miles. Move out 6.1 ft.
Draw a figure.



Make a plan. Rather than dive in and crunch numbers, let's do a bit of algebra for a second.
Carry out the plan. Let R be the beginning radius, and let q be the distance added to the radius. For the smaller orbit, the area enclosed was πR^2. For the new orbit, the area enclosed is $\pi(R + q)^2$. The difference is the new area enclosed, or

$$\text{New area} = \pi(R + q)^2 - \pi R^2 = \pi(R^2 + 2Rq + q^2) - \pi R^2$$

$$= \pi(2Rq + q^2) = \pi q(2R + q)$$

$$= 3.14159\left(6.1 \text{ ft} \times \frac{\text{mile}}{5280 \text{ ft}}\right)\left[2(8347.26 \text{ mi}) + \left(6.1 \text{ ft} \times \frac{\text{mile}}{5280 \text{ ft}}\right)\right]$$

$$= 61 \text{ square miles}$$

Look back. Is this reasonable? How could this result be checked? Are the units right? Are the conversion factors right? If q went to zero, does the formula work? If R went to zero?

If you are not convinced that this algebraic approach is better, start right out with numbers and see how you do. Besides that, by solving this a general way, several avenues open up for checking the result.

EXAMPLE 4.9 Specializing; Specializing to Check Results

Specialization is our all-time favorite tool for problem solving. For checking results as the final step in problem solving, specialization is the tool of choice. Results are often known for special cases, and this offers a wonderful way to check a general algorithm. Another standard approach is to push the algorithm's parameters to their limits and assure that the algorithm fails gracefully or scoots to infinity if it's supposed to.

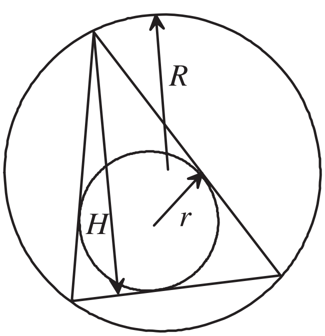
Problem Statement: In a triangle, let r be the radius of the inscribed circle, R the radius of the circumscribed circle, and H the longest altitude. Then prove

$$r + R \leq H$$

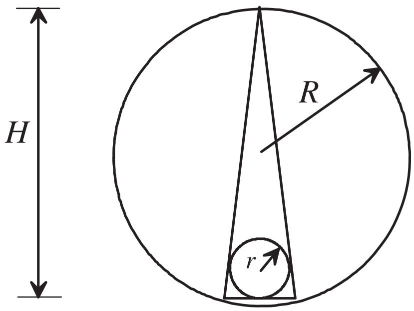
(In Polya, 1945, quoting from *The American Mathematical Monthly* 50, 1943, p. 124 and 51, 1944, pp. 234–236.)

Solution:

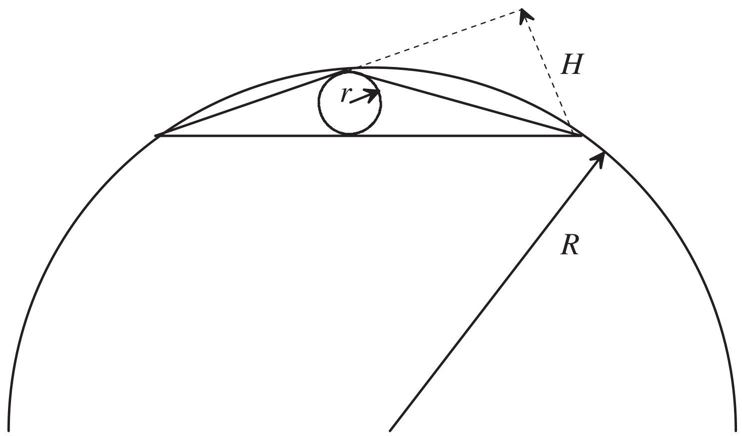
What is the unknown? How to show that the sum of the radii of inscribed and circumscribed circles is less than or equal to the greatest height of the triangle.
What is the given? The formula and directions for constructing the figure.
Draw a figure.



Devise a plan. This looks tough to prove. It's likely that $R + r$ is less than H in this example, but is it always true? Let's go to extremes. (That's where all the fun is anyway, right?) Let's look at a narrow-based, tall triangle and then a broad-based, short triangle.
Carry out the plan.



From the figure and with a bit of thought, we see that as the triangle gets taller, r will approach zero, and H will approach $2R$, so indeed H is greater than R. How about the other way?



But hang on a minute. Going this way, R is much larger than H, so the conjecture cannot be true.
Look back. Is there another way to prove that broad-based triangles don't work? Couldn't the tall, narrow triangle be turned over to become a short, broad triangle? What happens as the base gets infinitely long? When does $R + r = H$?

EXAMPLE 4.10 Decomposing and Recombining

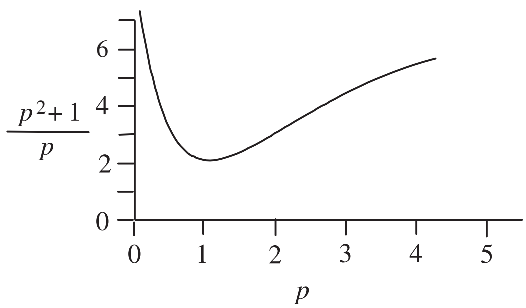
An engineer's most-used dictum is, Divide and conquer. Big problems are broken into little problems and solved one at a time, or parceled out among a group to be recombined for the finished product. This problem (from Schoenfeld, 1985) gives a sample of that approach.

Problem Statement: Given that $p, q, r,$ and s are real positive numbers, prove that

$$\frac{(p^2 + 1)(q^2 + 1)(r^2 + 1)(s^2 + 1)}{pqrs} \geq 16$$

Solution:

What is the unknown? How to show that the formula is always equal to or greater than 16.
What is the given? The formula.
Draw a figure. Don't know how to yet.
Devise a plan. As a first effort, we try numbers at random for $p, q, r,$ and s. If we can find a contradiction, we're done. But, no luck. In fact, usually the numbers are a lot bigger than 16. Hmmm . . . Isn't there symmetry in the formula? Really, isn't $(p^2 + 1)/p$ the same formula as $(q^2 + 1)/q$, $(r^2 + 1)/r$, and $(s^2 + 1)/s$? Then, if I can solve this small piece, I understand the whole formula. Now,
Carry out the plan.
Draw a figure.



Look back. Now it is clear how the formula works. Each piece will never be smaller than 2, and because $2 \times 2 \times 2 \times 2 = 16$, this is the minimum for the formula as a whole. How could this be checked? What happens at zero, the smallest positive number put into the formula? Can you prove that 2 is a minimum value for the formula?

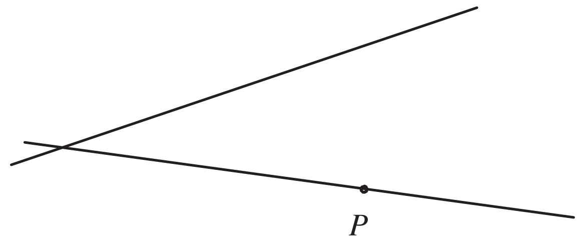
EXAMPLE 4.11 Taking the Problem as Solved

While taking the abstract approach is often useful (see the Inventor's Paradox, Example 4.8), the brain usually works better on concrete objects. If stuck on a problem, try to visualize the answer, no matter how approximate that vision may be. From this vision, try to work backward or deduce what is missing between the starting point and the solution.

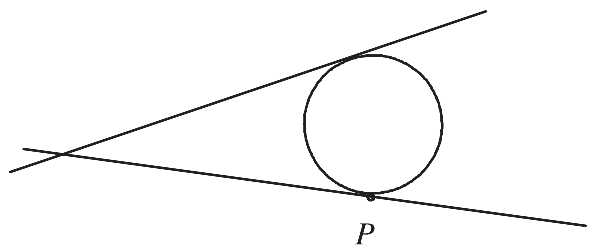
Problem Statement: Given two intersecting straight lines and a point P marked on one of these lines, and using a straightedge and a compass, construct a circle that is tangent to both lines and passes through point P as one of its points of tangency. (From Schoenfeld, 1985.)

Solution:

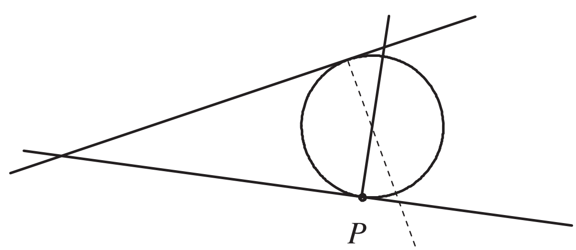
What is the unknown? How to construct the circle.
What is the given? Two intersecting lines and one point of tangency.
Draw a figure.



Devise a plan. Let's sketch in something that's close to a circle and close to the right size and see what happens.

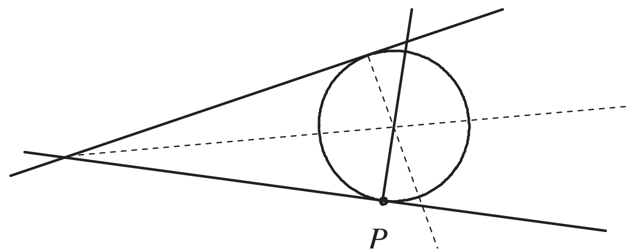


Well, not too bad for a sketch, but now what? The center of the circle looks straight up from point *P*. But it shouldn't be, right? Aren't tangents at right angles to the radius of the circle at the point of tangency? Well, that helps. I can construct a perpendicular at point *P*, but I still can't draw the circle because I don't know where the center is. Hmmm . . .



If I only knew where to place the other perpendicular from the other intersecting line (the dotted line in the figure), I'd be done. But I don't. Bummer. . . . But wait a minute! If that line *were* there, then the radius from the center to point *P* is the same distance as the radius to the tangency on the other line, right?

That means that the center of the circle is on the angular bisector, thus



So now the scheme is clear. At point *P*, construct a perpendicular. Bisect the angle between the two intersecting lines. The point where the bisector and perpendicular meet is the center of the circle.

Look back. Is this true for any two intersecting lines? How about obtuse angles? Can we do that? How about as the lines intersect at angles near zero (i.e., as the lines approach being parallel)?

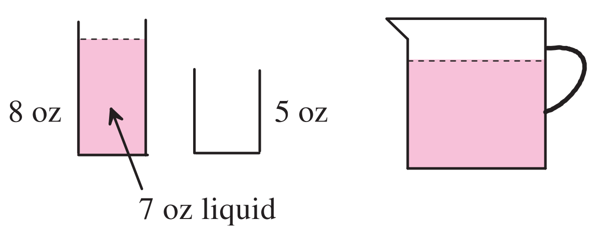
EXAMPLE 4.12 Working Forward / Working Backward

One of our students informed us that learning this trick alone was worth all the effort of the entire course. His work is synthesizing organic compounds, and this is his standard trick. He looks at what he wants to make and then mentally breaks the compound down to simpler components that he can make or, better yet, has sitting on his shelf. Another less esoteric application of this method is in solving maze puzzles. Start at the end and work back toward the start. Often the puzzle is easier this way. What usually happens in practice is that one starts from the beginning and works forward until stymied, then starts at the end and works backward. With luck one meets oneself somewhere in the middle.

Problem Statement: Measure exactly 7 ounces of liquid from a large container using only a 5-ounce container and an 8-ounce container. (Practiced with libations at many student hangouts.) (From Polya, 1945, and many others.)

Solution:

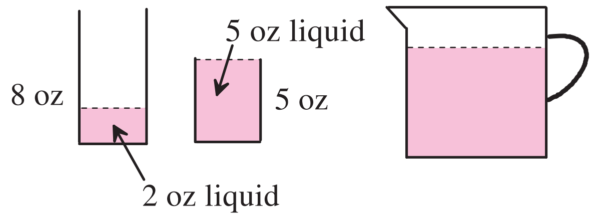
What is the unknown? How to measure seven ounces.
What is the given? Unlimited liquid, 5-oz and 8-oz containers.
Draw a figure.



This is, of course, the final state we seek: the 7 oz of liquid in the 8-oz container. (It's hard to put that much liquid into the 5-oz container.)

Devise a plan. The obvious comes to mind first. We can get 3 oz by filling the 8-oz container and then pouring from it into the 5-oz container until the 5-oz is full, leaving 3 oz of liquid in the 8-oz container. Also, we can get 2 oz by filling the 5-oz container, pouring it into the 8-oz container, filling the 5-oz container again, and emptying it into the 8-oz container until the 8-oz container is full, leaving 2 oz in the 5-oz container. Well, so what? Let's start at the end and see if we can meet in the middle.

Carry out the plan. If we back up one step from the final state we would have:



Bingo! I know how to get 2 oz. So now we perform the forward sequence to get 2 oz, then empty the full 8-oz container, pour in the 2 oz, refill the 5-oz container, and empty it into the 8-oz container.

Look back. Is each step correct? Anything special about 8-, 7-, and 5-oz measurements?

EXAMPLE 4.13 Argue by Contradiction; Using *Reductio ad Absurdum*

Occasionally, nature is kind enough to present a problem that must have only one of a small number of answers. It may be impossible to prove directly that one particular answer is correct. However, if you can prove that all others are not the answer, you prove indirectly that the remaining choice is correct.

Problem Statement: Write numbers using each of the 10 digits (0 through 9) exactly once such that the sum of the integers is exactly 100. For example, 29 + 10 + 38 + 7 + 6 + 5 + 4 = 99 uses each digit once, but sums to 99 instead of 100. And then there's 19 + 28 + 31 + 7 + 6 + 5 + 4 = 100, but . . . no zero. (From Polya, 1945.)

Solution:

What is the unknown? How to create numbers using the digits 0 through 9 so that the sum is 100.

What is the given? The digits 0 through 9, examples.

Draw a figure. For once, this doesn't help.

Devise a plan. After a lot of trial and error, one is convinced that the answer is not obvious, if it exists. However, proof that it is impossible is not obvious either. We'll bet it has something to do with which digits are in the tens place. Let's analyze how the sums can be made and, assuming it can be done, look for a contradiction.

Carry out the plan. First, the sum of the 10 digits, each taken singly, is:

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

If we add "0" to any of the digits, for example we change 1 to 10, the sum increases by 10 times whichever digit we used, less the digit itself. For 10, the sum is $45 + 10 - 1 = 54$. For 40, the sum is $45 + 40 - 4 = 81$. For 60, the sum is 99; for 70 the sum is 108, too large, as are 80 and 90. If digits are combined with numbers other than 0, for example combining the 4 and the 2 to make 42, the sum is $45 - 4 + 40 = 81$. Ahhh, only the digits used in the tens place are "lost" in the sum. We don't know how many digits will be lost (at most three, right?), but we can write an equation about their sum:

$$10 \times (\text{tens\_sum}) + 45 - \text{tens\_sum} = 100$$

Now we're cooking. But wait—solving the equation for tens_sum, we get 55/9? What kind of junk is this? Exactly the kind of junk we hoped for. Because there is no way to add integers and get 55/9, then the conjecture that the digits can sum to 100 must be false.

Look back. Is there some way to prove our method is correct? How about if we relax the requirement that the digits be integers? Can we use the digits and get 100?

4.7 SUMMARY

During the 40 or so years you will practice engineering, technology will keep changing dramatically. Therefore, it is impossible for your professors to teach you every fact you will need; today's state-of-the-art fact will become obsolete tomorrow. Instead, to prepare you for the future, we can help train your mind to think and solve problems. These skills never become obsolete.

Engineers face a variety of problems, including research, troubleshooting, mathematics, resource, and design problems. The problem-solving process can start once the problem is identified. To solve problems, engineers apply both synthesis (where pieces are combined together into a whole) and analysis (where the whole is dissected into pieces).

During your schooling, you will be confronted with thousands of problems. So you can check your work, the solution to these problems is provided to you. If you make a mistake, the consequences are not severe—the loss of a few points on a homework assignment or exam. In the real world, there are no answers in the back of the book, and the consequences of making a mistake can be catastrophic. You are well-advised to develop a systematic problem-solving strategy that leads you to the correct answer. We offer a suggested approach in the section on "error-free" problem solving.

One of the most valuable skills in engineering is the ability to estimate answers from incomplete information. Because we almost never have all the information we need to precisely solve a problem, nearly every engineering problem can be viewed as an estimation problem; they differ only in the degree of uncertainty in the final answer.

Many authors agree that problem solving can be broken into four or five steps: (1) understand the problem; (2) think about it; (3) devise a plan; (4) execute the plan; and (5) check your work. While thinking about the problem, a number of **heuristic** approaches (i.e., hints) can lead you to the solution. For example, you can exploit analogies, introduce auxiliary elements, generalize, specialize, decompose, take the problem as solved, work forward/backward, or argue by contradiction.

Perhaps you enjoy solving puzzles in the magazines or online. Although these may seem like trivial games, these puzzles can actually train your mind to solve engineering problems. Just as a boxer may train with a jump rope for an upcoming fight, an engineer can train with puzzles to prepare for the main event: solving engineering problems.

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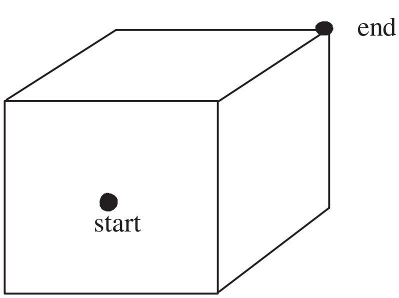
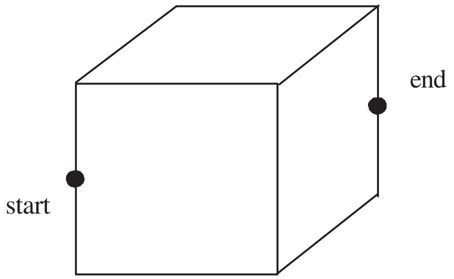
PROBLEMS

4.1 Mary Mermaid is taking swimming lessons in a circular pool. Mary starts at one edge of the pool and swims in a straight line for 12 meters, where she hits the edge of the pool. She turns and swims another 5.0 meters and again hits the edge of the pool. As she is examining her various scrapes, she realizes that she is exactly on the opposite side of the pool from where she started. What is the diameter of the pool?

4.2 What is the shortest path for an ant crawling on the surface of a unit cube from the starting point to the ending point shown:



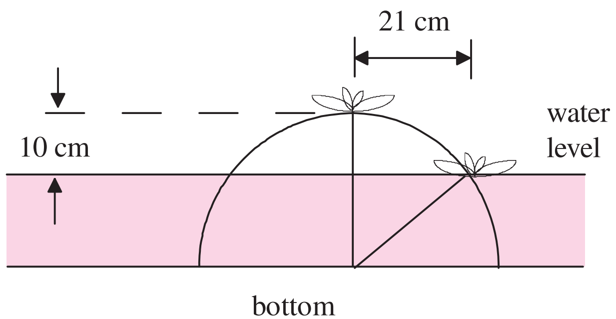
4.3 What is the shortest path for an ant crawling on the surface of a unit cube from the starting point to the ending point shown:



4.4 A string is fit snugly around the circumference of a spherical hot air balloon. More hot air is added (probably by a prominent scientist's lecture) and it now takes an additional 12.4 feet of string to fit around the circumference. What is the increase in diameter?

4.5 The poet Henry Wadsworth Longfellow, in his novel *Kavenaugh,* presented the following puzzle:

When a water lily stem is vertical, the blossom is 10 cm above the water. If the blossom is pulled to the right keeping the stem straight, the blossom touches the water 21 cm from where the stem came through the water when vertical. How deep is the water? (*Hint:* See figure.)

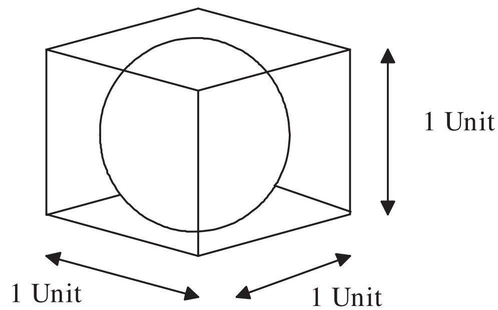


4.6 A farmer goes to market with \$100 to spend. Cows cost \$10 each, pigs cost \$3 each, and sheep cost half a dollar each. The farmer buys from the cattle dealer, the pig dealer, and the sheep dealer. She spends exactly \$100 and buys exactly 100 animals. How many of each animal does she buy?

4.7 The king and his two children are imprisoned at the top of a tall tower. Stone masons have been working on the tower and have left a pulley fixed at the top. Over the pulley runs a rope with a basket attached to either end. In the basket on the ground is a stone like the ones used to build the tower. The stone weighs 35 kg_f (75 lb_f). The king figures out that the stone can be used as a counterbalance—provided that the weight in either basket does not differ by more than 7 kg_f (15 lb_f). The king weighs 91 kg_f (195 lb_f), the princess weighs 49 kg_f (105 lb_f), and the prince weighs 42 kg_f (90 lb_f). How can they all escape from the tower? (They can throw the stone from the tower to the ground!) (Attributed to Lewis Carroll.)

4.8 From Problem 4.7, add a pig that weighs 28 kg_f (60 lb_f), a dog that weighs 21 kg_f (45 lb_f), and a cat that weighs 14 kg_f (30 lb_f). There is an extra limitation: there must be one human at the top and bottom of the tower to put the animals in and out of the basket. How can all six escape?

4.9 Calculate the ratio of the area to the volume for a unit cube, a unit sphere inscribed inside the cube, and a right cylinder inscribed inside the cube.



Next, for each having a unit volume (i.e., all three solids have the same volume) calculate the area-to-volume ratios for a sphere, cube, and cylinder.

4.10 Dr. Bogus, a close friend of ours, during a long session at the Dixie Chicken restaurant, was doodling on a napkin and "proved" that all numbers are equal. This came as quite a surprise to us, and we have delayed his calling the president only until you can review his proof. Here is a translation of the napkin doodles:

Pick two different numbers, *a* and *b,* and a nonzero number *c* that is the difference between *a* and *b,* thus:

$$a = b + c \qquad c \neq 0 \tag{1}$$

Multiply both sides by $a - b$

$$a(a - b) = (b + c)(a - b) \tag{2}$$

or

$$a^2 - ab = ab - b^2 + ac - bc \tag{3}$$

Subtract *ac* from both sides,

$$a^2 - ab - ac = ab - b^2 - bc \tag{4}$$

Factor out *a* from the left side of the equation, and *b* from the right,

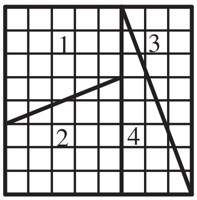
$$a\,(a - b - c) = b\,(a - b - c) \tag{5}$$

Eliminate the common factor from both sides, and

$$a = b \tag{6}$$

Because *a* and *b* can be any number, *c* can be positive or negative; thus, all numbers are equal to each other. Exactly which step(s) are wrong in the proof above, and why?

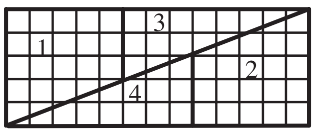
4.11 Dr. Bogus, now on a roll, also presented us with a proof that all our attempts to measure area are wrong. Again, we were somewhat startled, but he showed us that by simply rearranging the space within an area, one could get different answers. He presented us with the following drawing:



A square 8 units on a side is cut into 4 pieces. The pieces are then rearranged into a 5×13 rectangle.

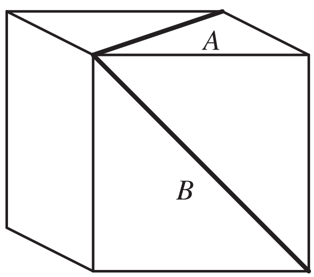
Wait a minute!

$8 \times 8 = 64$
$5 \times$

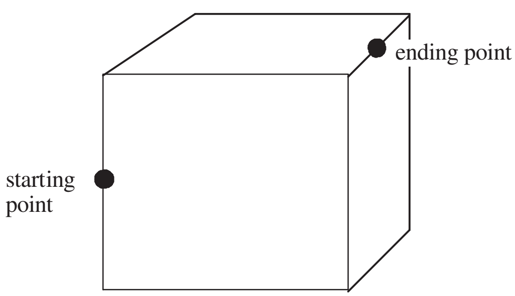


Where does the extra square come from? Help us, or Dr. Bogus will get a Nobel Prize before we do.

4.12 What is the plane angle between Line *A* and Line *B,* drawn on a unit cube?



4.13 A fly is at the midpoint of the front edge of a unit cube as shown in the figure. What minimum distance must it crawl to arrive at the midpoint of the opposite top edge?

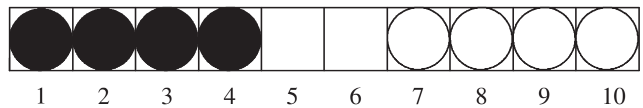


4.14 Suppose you wish to average 40 mph on a trip and find that when you are half the distance to your destination you have averaged 30 mph. How fast should you travel in the remaining half of the trip to attain an overall average of 40 mph?

4.15 A ship leaves port at 12:00 P.M. (noon) and sails east at 10 miles per hour. Another leaves the same port at 1:00 P.M. and sails north at 20 miles per hour. At what time are the ships 50 miles apart?

4.16 At a certain Point *A* on level ground, the angle of elevation to the top of a tower is observed to be 33°. At another Point *B,* in line with *A* and the base of the tower and 50 feet closer to the tower, the angle of elevation to the top is observed to be 68°. Find the height of the tower.

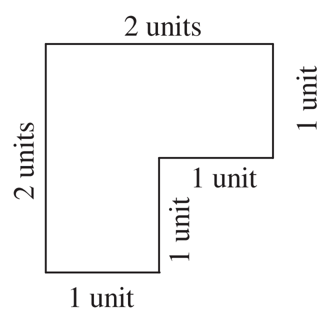
4.17 Using a straight track with 10 spots, set four pennies in the leftmost spots and four dimes in the rightmost spots, leaving the center two spots blank, as shown in the figure:



The object of the game is to move the pennies to the rightmost spots and the dimes to the left. You can move coins only by jumping a single coin into a vacant spot or by moving forward one to a vacant spot. No backward moves are allowed: all pennies must move to the right; all dimes to the left. Number the squares as shown to describe your solution.

4.18 Jack Jokely got up late one morning and was trying to find a clean pair of socks. He has a total of six pairs (three maroon and three white) in his drawer. Jack is not a very good housekeeper and his socks were just thrown in the drawer and, to make the morning perfect, the lightbulb is burned out so he can't see. How many socks does he have to pull out of the drawer before he gets a matched pair (two maroon socks or two white socks)?

4.19 A farmer has a piece of land he wants to give to his four sons (see figure below). The land must be divided into four equal-sized and equal-shaped pieces. How can the farmer accomplish his task?



4.20 A peasant had the wonderful good fortune to save the life of the King of Siam. The King, in turn, granted her payment in any form she wished. "I am a simple woman," said the peasant; "I wish merely to never be hungry again." The peasant asked to have payment made onto a huge chessboard painted on the floor. Payment was to be as follows: one grain of rice on the first square, two grains on the second square, four grains on the third square, eight onto the fourth, etc., until all 64 squares were filled. How many grains of rice will the peasant receive from the King? Roughly what volume of rice is this?

4.21 In a field there are cows, birds, and spiders. Spiders have four eyes and eight legs each. In the field there are 20 eyes and 30 legs. All three animals are present, and there is an odd number of each animal. How many spiders, cows, and birds are present? (From Gardner, 1978.)

4.22 Florence Fleetfoot and Larry Lethargy race on a windy day: 100 yards with the wind, they instantly turn around, and race 100 yards back against the wind. Larry is unaffected by the wind, but Florence goes only 90% as fast running against the wind as when running with no wind. Running with the wind improves Florence's no-wind speed by 10%. On a day with no wind, Florence and Larry tie in a 100-yard race. Who wins the windy-day race, and by how much?

4.23 Given a fixed triangle *T* with base *B,* show that it is always possible to construct with a straightedge and compass, a straight line parallel to *B* dividing the triangle into two parts of equal area. (From Schoenfeld, 1985.)

4.24 A pipe one mile in length and with 1-inch inside diameter is set in the ground such that it is optically straight, that is, a laser beam will pass down the centerline of the whole pipe. The centerline of the pipe is exactly level, that is, it is at right angles with a line passing through the center of the earth. Now, by means of a funnel and rubber tubing, water is poured into one end of the pipe until it flows from the other end. The funnel and tubing are removed and water is allowed to flow freely from the open ends. Neglect the surface tension of the water (it has little effect on this problem anyway) and calculate how much water is left in the pipe (say to within 10%). (*Hint:* The answer is not zero, and take the radius of the earth to be exactly 4000 miles should you need that information. *Further hint:* This problem is tougher than the rest of 'em put together.)

4.25 Estimate the number of toothpicks that can be made from a log measuring 3 ft in diameter and 20 ft long.

4.26 Estimate the number of drops in the ocean. Compare your estimate to Avogadro's number.

4.27 Estimate the maximum number of cars per hour that can travel down two lanes of a highway as a function of speed (i.e., at 50, 60, and 70 mph). For safety reasons, the cars must be spaced one car length apart for each 10 mph they are traveling. For example, if traffic is moving at 50 mph, then there must be five car lengths between each car. A city councilman proposes to solve traffic congestion by increasing the speed limit. How would you respond to this proposal?

4.28 If electricity costs \$0.07/kWh (kilowatt-hour), how much money does a typical household spend for electricity each year?

4.29 Estimate the number of books in your university's main library.

4.30 Estimate the amount of garbage produced by the United States each year.

4.31 Estimate the amount of gasoline consumed by automobiles in the United States each year. If this gasoline were stored in a single tank measuring 0.1 mi by 0.1 mi at the base, what is its height?

4.32 A vertical 10-ft column must support a 1,000,000-lb_f load located at the top of the column. An engineer must decide whether to construct it from concrete or steel. She will select the column that is lightest. Assume the column will be designed with a safety factor of 3, meaning the constructed column could support a load three times heavier, but no more. Estimate the mass of a steel column and a concrete column.

4.33 Estimate the number of party balloons it would take to fill your university's football stadium to the top.

4.34 Estimate the amount of money students at your university spend on fast food each semester.

4.35 Estimate the mass of the air on planet earth. What fraction of the total earth mass is air?

4.36 Estimate the mass of water on planet earth. What fraction of the total earth mass is water?

4.37 Estimate the length of time it would take for a passenger jet flying at Mach 0.8 ($\frac{8}{10}$ the speed of sound) to fly around the world. Make allowances for refueling.

4.38 Using Archimedes' principle, estimate the mass that can be lifted by a balloon measuring 30 ft in diameter. The temperature of the air in the balloon is 70°C, and its pressure is 1 atm.

Glossary

application A process whereby appropriate information is identified for the problem at hand.

Archimedes' principle The total mass of a floating object equals the mass of the fluid displaced by the object.

comprehension The step in which the proper theory and data are used to actually solve the problem.

conversion factor A numerical factor that, through multiplication or division, converts a quantity expressed in one system of units to another system of units.

engineering analysis paper A paper with a light grid pattern that can be used for graphing or drafting.

heuristic The use of speculative strategies for solving a problem.

iterative procedure Repetition of a sequence of steps to solve a problem.

manipulative models A class of problems that have some physical entity that can be used to one's advantage in order to solve a problem.

parallelepiped A solid with six parallelogram faces that are parallel to the opposite face.

problem solving A process in which an individual or a team applies knowledge, skills, and understanding to achieve a desired outcome in an unfamiliar situation.

Pythagorean theorem The sum of the squares of the lengths of the sides of a right triangle is equal to the square of the length of the hypotenuse.

reductionism The ability to logically break down a problem into pieces.