Optimal amount of a public good

In competitive markets for private goods, the optimal quantity of the good occurs where the marginal value of the good is equal to its marginal cost of production. Is there a similar rule for public goods?

Suppose there are N consumers in an economy, each of whom derives some value from the provision of a public good. For individual i, let this value be given by $V_i(G)$ where G is the amount of the public good. Notice that the *amount* of the public good is the same for each of our N individuals because such goods are nonrival, but the *value* each receives from this amount may vary. Your valuation of the amount of national defense provided and your neighbor's valuation may differ significantly, even if your taxes are the same!

The total value of the public good produced, $V(G)$, is the sum of all the individual valuations:

$$V(G) = \sum_{i=1}^{N} V_i(G) .$$

Let the cost of providing a particular level G be given by the function $C(G)$. $C'(G) > 0$ is its marginal cost. The net benefit to society of any specific value of G is then simply the difference between the total value and the cost: $NB(G) = V(G) - C(G) = \sum_{i=1}^{N} V_i(G) - C(G)$. From society's standpoint, the optimal value of G is the one that maximizes this net benefit.

To maximize $NB(G)$, we take its first derivative with respect to G and equate it to zero, making use of the fact that the derivative of a sum is equal to the sum of the derivatives: $NB'(G) = \sum_{i=1}^{N} \frac{\mathrm{d}V_i(G)}{\mathrm{d}G} - C'(G) = 0$, or in words, the sum of all consumers' marginal valuations of the good should equal its marginal cost of production.

Compare this to the result for a private good, Q, from which the i^{th} consumer receives a value of $V_i(Q_i)$. Since a private good is rival by definition, total output is $Q = \sum_{i=1}^{N} Q_i$ and Q_i may be different for each consumer. The total benefit received by the public is $V(Q) = \sum_{i=1}^{N} V_i(Q_i)$ and its total cost is $C(Q)$. Once again, the optimal output is found at the maximum net benefit, or difference between value and cost. Differentiating with respect to Q_i, we require that $\frac{\mathrm{d}V_i}{\mathrm{d}Q_i} - C'(Q)\frac{\mathrm{d}Q}{\mathrm{d}Q_i} = 0$. Since $\frac{\mathrm{d}Q}{\mathrm{d}Q_i} = 1$, this implies that the marginal valuation of each consumer of his or her consumption level equals the cost of producing the very last unit, $C'(Q)$.