Utility-maximizing rule

Suppose that a consumer's preferences can be represented by the utility function $U = f(X, Y)$ where X and Y represent the amounts of goods X and Y consumed. Our consumer's problem is to maximize utility subject to the constraint that total expenditures on the two goods equals income: $I = P_X X + P_Y Y$.

To solve this problem, we form the function $V = f(X, Y) + \lambda(I - P_X X - P_Y Y)$. (This function is called a "Lagrangian" and the variable λ is known as the "Lagrange multiplier"). In this specification, we can see that if the values of X and Y satisfy the constraint, $I - P_X X - P_Y Y = 0$ and the values of V and U are the same. Comparing V and U, then, it is clear that as long as λ is not equal to zero, maximizing V is equivalent to maximizing U subject to the income constraint.)

To maximize utility, all the partial derivatives of this Lagrange equation must be set to zero:

$$\partial V/\partial X = \partial f/\partial X - \lambda P_X = 0$$
$$\partial V/\partial Y = \partial f/\partial Y - \lambda P_Y = 0$$
$$\partial V/\partial \lambda = I - P_X X - P_Y Y = 0$$

(Note that this condition fulfills the requirement imposed by the budget constraint.)

Assuming that the second-order conditions for a maximum are fulfilled, we can solve these three equations for the utility-maximizing values of X, Y, and λ. The first equation can be solved for λ to obtain $\lambda = \dfrac{\partial f / \partial X}{P_X}$ and from the second equation, we see that $\lambda = \dfrac{\partial f / \partial Y}{P_X}$. Since both of these expressions equal λ, they must equal each other: $\dfrac{\partial f / \partial X}{P_X} = \dfrac{\partial f / \partial Y}{P_X}$. The terms in the numerators are the marginal utilities of goods X and Y, respectively, so this is the condition that we require. Maximum utility is achieved when the marginal utility per dollar of each good is the same: $\dfrac{MU_X}{P_X} = \dfrac{MU_Y}{P_Y}$. Of course, the final equation implies that the consumer's budget must be exhausted as well.

Alternatively, the first two equations could be solved as follows. Add λP_X to both sides of the first equation and λP_Y to both sides of the second to obtain $\partial f/\partial X = \lambda P_X$ and $\partial f/\partial Y = \lambda P_Y$. Next, divide the first of these equalities by the second to obtain $\dfrac{\partial f / \partial X}{\partial f / \partial Y} = \dfrac{\lambda P_X}{\lambda P_Y} = \dfrac{P_X}{P_Y}$. The term on the left, the ratio of the marginal utilities, is known as the "marginal rate of substitution," the rate at which a consumer is willing to substitute X for Y. In this formulation, we see that maximum utility requires that the consumer's marginal rate of substitution be equal to the ratio of the prices of the two goods. That is, a consumer will adjust purchases of the two goods until her willingness to trade one for the other just matches the rate at which they can be traded in the marketplace, as given by the ratio of their prices.