

A P P E N D I X

A

LINEAR ALGEBRA AND
COMPLEX NUMBERS**A.1 SOLVING SIMULTANEOUS LINEAR
EQUATIONS, CRAMER'S RULE,
AND MATRIX EQUATION**

The solution of simultaneous equations, such as those that are often seen in circuit theory, may be obtained relatively easily by using Cramer's rule. This method applies to 2×2 or larger systems of equations. Cramer's rule requires the use of the concept of determinant. Linear, or matrix, algebra is valuable because it is systematic, general, and useful in solving complicated problems. A determinant is a scalar defined on a square array of numbers, or matrix, such as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad (\text{A.1})$$

In this case the matrix is a 2×2 array with two rows and two columns, and its determinant is defined as

$$\det = a_{11}a_{22} - a_{12}a_{21} \quad (\text{A.2})$$

A third-order, or 3×3 , determinant such as

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad (\text{A.3})$$

is given by

$$\begin{aligned} \det &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned} \quad (\text{A.4})$$

For higher-order determinants, you may refer to a linear algebra book. To illustrate Cramer's method, a set of two equations in general form will be solved here. A set of two linear simultaneous algebraic equations in two unknowns can be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \quad (\text{A.5})$$

where x_1 and x_2 are the two unknowns. The coefficients a_{11} , a_{12} , a_{21} , and a_{22} are known quantities. The two quantities on the right-hand sides, b_1 and b_2 , are also known (these are typically the source currents and voltages in a circuit problem). The set of equations can be arranged in matrix form, as shown in equation A.6.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (\text{A.6})$$

In equation A.6, a coefficient matrix multiplied by a vector of unknown variables is equated to a right-hand-side vector. Cramer's rule can then be applied to find x_1 and x_2 , using the following formulas:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad (\text{A.7})$$

Thus, the solution is given by the ratio of two determinants: the denominator is the determinant of the matrix of coefficients, while the numerator is the determinant of the same matrix with the right-hand-side vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in this case) substituted in place of the column of the coefficient matrix corresponding to the desired variable (i.e., first column for x_1 , second column for x_2 , etc.). In a circuit analysis problem, the coefficient matrix is formed by the resistance (or conductance) values, the vector of unknowns is composed of the mesh currents (or node voltages), and the right-hand-side vector contains the source currents or voltages.

In practice, many calculations involve solving higher-order systems of linear equations. Therefore, a variety of computer software packages are often used to solve higher-order systems of linear equations.

CHECK YOUR UNDERSTANDING

A.1 Use Cramer's rule to solve the system

$$5v_1 + 4v_2 = 6$$

$$3v_1 + 2v_2 = 4$$

A.2 Use Cramer's rule to solve the system

$$\begin{aligned} i_1 + 2i_2 + i_3 &= 6 \\ i_1 + i_2 - 2i_3 &= 1 \\ i_1 - i_2 + i_3 &= 0 \end{aligned}$$

A.3 Convert the following system of linear equations into a matrix equation as shown in equation A.6, and find matrices A and b .

$$\begin{aligned} 2i_1 - 2i_2 + 3i_3 &= -10 \\ -3i_1 + 3i_2 - 2i_3 + i_4 &= -2 \\ 5i_1 - i_2 + 4i_3 - 4i_4 &= 4 \\ i_1 - 4i_2 + i_3 + 2i_4 &= 0 \end{aligned}$$

Answers: $v_1 = 2, v_2 = -1, v_3 = 2, v_4 = 1$;

$$A = \begin{bmatrix} 2 & -2 & 3 & 0 \\ -3 & 3 & -2 & 1 \\ 5 & -1 & 4 & -4 \\ 1 & -4 & 1 & 2 \end{bmatrix}, b = \begin{bmatrix} -10 \\ -2 \\ 4 \\ 0 \end{bmatrix}$$

A.2 INTRODUCTION TO COMPLEX ALGEBRA

From your earliest training in arithmetic, you have dealt with real numbers such as 4, -2 , $\frac{5}{9}$, π , e , and so on, which may be used to measure distances in one direction or another from a fixed point. However, a number that satisfies the equation

$$x^2 + 9 = 0 \tag{A.8}$$

is not a real number. Imaginary numbers were introduced to solve equations such as equation A.8. Imaginary numbers add a new dimension to our number system. To deal with imaginary numbers, a new element, j , is added to the number system having the property

$$j^2 = -1 \tag{A.9}$$

or

$$j = \sqrt{-1} \tag{A.11}$$

Thus, we have $j^3 = -j$, $j^4 = 1$, $j^5 = j$, and so on. Using equation A.9, you can see that the solutions to equation A.8 are $\pm j3$. In mathematics, the symbol i is used for the imaginary unit, but this might be confused with current in electrical engineering. Therefore, the symbol j is used in this book.

A complex number (indicated in boldface notation) is an expression of the form

$$\mathbf{A} = a + jb \tag{A.10}$$

where a and b are real numbers. The complex number \mathbf{A} has a real part a and an imaginary part b , which can be expressed as

$$\begin{aligned} a &= \text{Re } \mathbf{A} \\ b &= \text{Im } \mathbf{A} \end{aligned} \tag{A.11}$$

It is important to note that a and b are both real numbers. The complex number $a + jb$ can be represented on a rectangular coordinate plane, called the *complex plane*, by interpreting it as a point (a, b) . That is, the horizontal coordinate is a in real axis, and the vertical coordinate is b in imaginary axis, as shown in Figure A.1. The complex number $\mathbf{A} = a + jb$ can also be

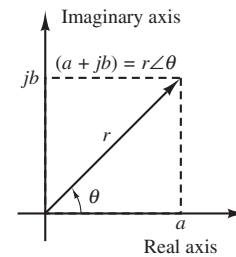


Figure A.1 Polar form representation of complex numbers

uniquely located in the complex plane by specifying the distance r along a straight line from the origin and the angle θ , which this line makes with the real axis, as shown in Figure A.1. From the right triangle of Figure A.1, we can see that

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1} \left(\frac{b}{a} \right) \\ a &= r \cos \theta \\ b &= r \sin \theta \end{aligned} \tag{A.12}$$

Then we can represent a complex number by the expression

$$\mathbf{A} = r e^{j\theta} = r \angle \theta \tag{A.13}$$

which is called the polar form of the complex number. The number r is called the magnitude (or amplitude), and the number θ is called the angle (or argument). The two numbers are usually denoted by $r = |\mathbf{A}|$ and $\theta = \arg \mathbf{A} = \angle \mathbf{A}$.

Given a complex number $\mathbf{A} = a + jb$, the *complex conjugate* of \mathbf{A} , denoted by the symbol \mathbf{A}^* , is defined by the following equalities:

$$\begin{aligned} \operatorname{Re} \mathbf{A}^* &= \operatorname{Re} \mathbf{A} \\ \operatorname{Im} \mathbf{A}^* &= -\operatorname{Im} \mathbf{A} \end{aligned} \tag{A.14}$$

That is, the sign of the imaginary part is reversed in the complex conjugate.

Finally, we should remark that two complex numbers are equal *if and only if* the real parts are equal and the imaginary parts are equal. This is equivalent to stating that two complex numbers are equal only if their magnitudes are equal and their arguments are equal.

The following examples and exercises should help clarify these explanations.

EXAMPLE A.1

Convert the complex number $\mathbf{A} = 3 + j4$ to its polar form.

Solution:

$$\begin{aligned} r &= \sqrt{3^2 + 4^2} = 5 & \theta &= \tan^{-1} \left(\frac{4}{3} \right) = 53.13^\circ \\ \mathbf{A} &= 5 \angle 53.13^\circ \end{aligned}$$

EXAMPLE A.2

Convert the number $\mathbf{A} = 4 \angle (-60^\circ)$ to its complex form.

Solution:

$$\begin{aligned} a &= 4 \cos(-60^\circ) = 4 \cos(60^\circ) = 2 \\ b &= 4 \sin(-60^\circ) = -4 \sin(60^\circ) = -2\sqrt{3} \end{aligned}$$

Thus, $\mathbf{A} = 2 - j2\sqrt{3}$

Addition and subtraction of complex numbers take place according to the following rules:

$$\begin{aligned}(a_1 + jb_1) + (a_2 + jb_2) &= (a_1 + a_2) + j(b_1 + b_2) \\ (a_1 + jb_1) - (a_2 + jb_2) &= (a_1 - a_2) + j(b_1 - b_2)\end{aligned}\tag{A.15}$$

Multiplication of complex numbers in polar form follows the law of exponents. That is, the magnitude of the product is the product of the individual magnitudes, and the angle of the product is the sum of the individual angles, as shown below.

$$\mathbf{(A)(B)} = (Ae^{j\theta})(Be^{j\phi}) = AB e^{j(\theta+\phi)} = AB \angle(\theta + \phi)\tag{A.16}$$

If the numbers are given in rectangular form and the product is desired in rectangular form, it may be more convenient to perform the multiplication directly, using the rule that $j^2 = -1$, as illustrated in equation A.17.

$$\begin{aligned}(a_1 + jb_1)(a_2 + jb_2) &= a_1a_2 + ja_1b_2 + ja_2b_1 + j^2b_1b_2 \\ &= (a_1a_2 + j^2b_1b_2) + j(a_1b_2 + a_2b_1) \\ &= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)\end{aligned}\tag{A.17}$$

Division of complex numbers in polar form follows the law of exponents. That is, the magnitude of the quotient is the quotient of the magnitudes, and the angle of the quotient is the difference of the angles, as shown in equation A.18.

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{Ae^{j\theta}}{Be^{j\phi}} = \frac{A \angle \theta}{B \angle \phi} = \frac{A}{B} \angle(\theta - \phi)\tag{A.18}$$

Division in the rectangular form can be accomplished by multiplying the numerator and denominator by the complex conjugate of the denominator. Multiplying the denominator by its complex conjugate converts the denominator to a real number and simplifies division. This is shown in Example A.4. Powers and roots of a complex number in polar form follow the laws of exponents, as shown in equations A.19 and A.20.

$$\mathbf{A}^n = (Ae^{j\theta})^n = A^n e^{jn\theta} = A^n \angle n\theta\tag{A.19}$$

$$\begin{aligned}\mathbf{A}^{1/n} &= (Ae^{j\theta})^{1/n} = A^{1/n} e^{j\theta/n} \\ &= \sqrt[n]{A} \angle \left(\frac{\theta + k2\pi}{n} \right) \quad k = 0, \pm 1, \pm 2, \dots\end{aligned}\tag{A.20}$$

EXAMPLE A.3

Perform the following operations, given that $\mathbf{A} = 2 + j3$ and $\mathbf{B} = 5 - j4$.

- (a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - \mathbf{B}$ (c) $2\mathbf{A} + 3\mathbf{B}$

Solution:

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (2 + 5) + j[3 + (-4)] = 7 - j \\ \mathbf{A} - \mathbf{B} &= (2 - 5) + j[3 - (-4)] = -3 + j7\end{aligned}$$

For part (c), $2\mathbf{A} = 4 + j6$ and $3\mathbf{B} = 15 - j12$. Thus, $2\mathbf{A} + 3\mathbf{B} = (4 + 15) + j[6 + (-12)] = 19 - j6$

EXAMPLE A.4

Perform the following operations in both rectangular and polar form, given that $\mathbf{A} = 3 + j3$ and $\mathbf{B} = 1 + j\sqrt{3}$.

- (a)
- \mathbf{AB}
- (b)
- $\mathbf{A} \div \mathbf{B}$

Solution:

- (a) In rectangular form:

$$\begin{aligned}\mathbf{AB} &= (3 + j3)(1 + j\sqrt{3}) = 3 + j3\sqrt{3} + j3 + j^23\sqrt{3} \\ &= (3 + j^23\sqrt{3}) + j(3 + 3\sqrt{3}) \\ &= (3 - 3\sqrt{3}) + j(3 + 3\sqrt{3})\end{aligned}$$

To obtain the answer in polar form, we need to convert \mathbf{A} and \mathbf{B} to their polar forms:

$$\begin{aligned}\mathbf{A} &= 3\sqrt{2}e^{j45^\circ} = 3\sqrt{2}\angle 45^\circ \\ \mathbf{B} &= \sqrt{4}e^{j60^\circ} = 2\angle 60^\circ\end{aligned}$$

Then

$$\mathbf{AB} = (3\sqrt{2}e^{j45^\circ})(\sqrt{4}e^{j60^\circ}) = 6\sqrt{2}\angle 105^\circ$$

- (b) To find
- $\mathbf{A} \div \mathbf{B}$
- in rectangular form, we can multiply
- \mathbf{A}
- and
- \mathbf{B}
- by
- \mathbf{B}^*
- .

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{3 + j3}{1 + j\sqrt{3}} \frac{1 - j\sqrt{3}}{1 - j\sqrt{3}}$$

Then

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(3 + 3\sqrt{3}) + j(3 - 3\sqrt{3})}{4}$$

In polar form, the same operation may be performed as follows:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{3\sqrt{2}\angle 45^\circ}{2\angle 60^\circ} = \frac{3\sqrt{2}}{2}\angle(45^\circ - 60^\circ) = \frac{3\sqrt{2}}{2}\angle(-15^\circ)$$

Euler's Identity

This formula extends the usual definition of the exponential function to allow for complex numbers as arguments. Euler's identity states that

$$e^{j\theta} = \cos \theta + j \sin \theta \quad (\text{A.21})$$

All the standard trigonometry formulas in the complex plane are direct consequences of Euler's identity. The two important formulas are

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (\text{A.22})$$

EXAMPLE A.5

Using Euler's formula, show that

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Solution:

Using Euler's formula gives

$$e^{j\theta} = \cos \theta + j \sin \theta$$

Extending the above formula, we can obtain

$$e^{-j\theta} = \cos(-\theta) + j \sin(-\theta) = \cos \theta - j \sin \theta$$

Thus,

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

CHECK YOUR UNDERSTANDING

- A.4** In a certain AC circuit, $V = ZI$, where $Z = 7.75 \angle 90^\circ$ and $I = 2 \angle -45^\circ$. Find V .
- A.5** In a certain AC circuit, $V = ZI$, where $Z = 5 \angle 82^\circ$ and $V = 30 \angle 45^\circ$. Find I .
- A.6** Show that the polar form of AB in Example A.4 is equivalent to its rectangular form.
- A.7** Show that the polar form of $A \div B$ in Example A.4 is equivalent to its rectangular form.
- A.8** Using Euler's formula, show that $\sin \theta = (e^{j\theta} - e^{-j\theta})/2j$.

Answers: $V = 15.5 \angle 45^\circ$; $I = 6 \angle -37^\circ$

