APPENDIX 10A: Black–Scholes Option Pricing Model

In 1973, Fisher Black and Myron Scholes published their option pricing model.[26] Since its publication, improvements and extensions have been made to the model and it is now used by most professional option traders. To see how the Black–Scholes model works, we first look at how a European call option can be valued using a simple binomial model. Suppose a stock's price is currently \$60 and it is known that the price at the end of the month will be either \$66 or \$54. A call option on the stock has an exercise price of \$63 and a one-month maturity. The option involves 100 shares of the underlying stock. As illustrated in Figure 10–18, if the stock price is \$66 after one month, the value of the option is \$3 (\$66 − \$63). If the stock price is \$54, the option's value is 0 (since the exercise price is greater than \$54).

For a portfolio consisting of a long position in X shares of the stock, the value of the portfolio is $66X - 3$ if the stock price goes to \$66 and \$54X otherwise. When $X = .25$ (since the option involves 100 shares of the underlying stock, this means the portfolio consists of 25 shares of the underlying stock), the portfolio is riskless. That is, the portfolio's value is unaffected by the change in the stock price over the one month:

$$66X - 3 - 54X = 13.5$$

The value of the portfolio at the beginning of the month when $X = .25$ is:

$$60 \times .25 - \text{C} = 15 - C$$

where C is the value of the call option at the beginning of the month. If the risk-free rate of interest is 1/2 percent per month, then:

$$1.005(15 - C) = 13.5$$

or

$$C = 15 - (13.5/1.005) = 1.567$$

Or the value of the option at the beginning of the month must be \$1.567.

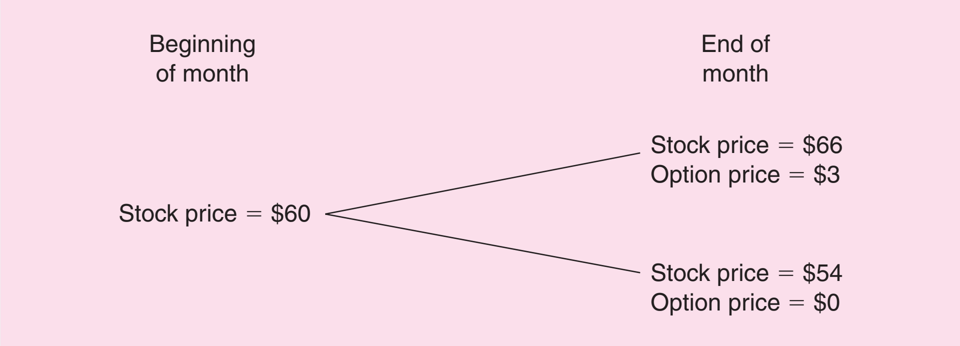
Extending beyond this very simple example, the Black–Scholes model uses historical stock price data to determine the exact value of a call option. Specifically, the Black–Scholes option pricing model used to value European options is presented in the following equation:

$$C = N(d_1)S - E(e^{-rT})N(d_2)$$

$$d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Figure 10–18 **Binomial Model of Stock Price Movements**



[26] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (May–June, 1973), pp. 637–54.

where

C = Call option price
S = Price on the asset underlying the option
E = Exercise price of the option
r = Riskless rate of interest over one year
σ = Standard deviation of the underlying asset's return
T = Time to expiration of the option as a fraction of one year
e = Base of the natural logarithm, or the exponential function
$\ln(S/E)$ = Natural log of S/E
$N(d)$ = Value of the cumulative normal distribution evaluated at d_1 and d_2

The Black–Scholes option pricing formula assumes the following:

- Capital markets are frictionless (i.e., there are no transaction costs or taxes and all information is simultaneously and freely available to all investors).
- The variability in the underlying asset's return is constant.
- The probability distribution of the underlying asset's price is log normal.
- The risk-free rate is constant and known over time.
- No dividends are paid on the underlying asset.
- No early exercise is allowed on the option.

Example 10-3 Using the Black–Scholes Formula to Value a Call Option

Suppose you own a call option on a stock for which the following applies:

Underlying stock′s price = \$60
Exercise price on the option = \$58
Annual risk-free rate = 5 percent
Time to expiration on the option = 3 months
Standard deviation of the underlying stock's return = .12

To calculate the value of the option, we first calculate d_1 and d_2 as follows:

$$d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln(60/58) + (.05 + (.12)^3/2)(3/12)}{.12(3/12)^{1/2}} = .8034$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$= .8034 - .12(3/12)^{1/2} = .7434$$

Next, the values of $N(d_1)$ and $N(d_2)$ are found from Table 10–11, which shows the cumulative normal distribution. Interpolation from the values in Table 10–11 give $N(d_1)$:

d_1	$N(d_1)$
.80	.7881
.8034	?
.85	.8023

or:

$$N(d_1) = .7891$$

and $N(d_2)$:

d_2	$N(d_2)$
.70	.7580
.7434	?
.75	.7734

TABLE 10–11 Values of the Cumulative Normal Distribution

d	$N(d)$	d	$N(d)$	d	$N(d)$	d	$N(d)$	d	$N(d)$	d	$N(d)$
		−2.00	.0228	−1.00	.1587	.00	.5000	1.00	.8413	2.00	.9773
−2.95	.0016	−1.95	.0256	−.95	.1711	.05	.5199	1.05	.8531	2.05	.9798
−2.90	.0019	−1.90	.0287	−.90	.1841	.10	.5398	1.10	.8643	2.10	.9821
−2.85	.0022	−1.85	.0322	−.85	.1977	.15	.5596	1.15	.8749	2.15	.9842
−2.80	.0026	−1.80	.0359	−.80	.2119	.20	.5793	1.20	.8849	2.20	.9861
−2.75	.0030	−1.75	.0401	−.75	.2266	.25	.5987	1.25	.8944	2.25	.9878
−2.70	.0035	−1.70	.0446	−.70	.2420	.30	.6179	1.30	.9032	2.30	.9893
−2.65	.0040	−1.65	.0495	−.65	.2578	.35	.6368	1.35	.9115	2.35	.9906
−2.60	.0047	−1.60	.0548	−.60	.2743	.40	.6554	1.40	.9192	2.40	.9918
−2.55	.0054	−1.55	.0606	−.55	.2912	.45	.6735	1.45	.9265	2.45	.9929
−2.50	.0062	−1.50	.0668	−.50	.3085	.50	.6915	1.50	.9332	2.50	.9938
−2.45	.0071	−1.45	.0735	−.45	.3264	.55	.7088	1.55	.9394	2.55	.9946
−2.40	.0082	−1.40	.0808	−.40	.3446	.60	.7257	1.60	.9459	2.60	.9953
−2.35	.0094	−1.35	.0855	−.35	.3632	.65	.7422	1.65	.9505	2.65	.9960
−2.30	.0107	−1.30	.0968	−.30	.3821	.70	.7580	1.70	.9554	2.70	.9965
−2.25	.0122	−1.25	.1057	−.25	.4013	.75	.7734	1.75	.9599	2.75	.9970
−2.20	.0139	−1.20	.1151	−.20	.4207	.80	.7881	1.80	.9641	2.80	.9974
−2.15	.0158	−1.15	.1251	−.15	.4404	.85	.8023	1.85	.9678	2.85	.9973
−2.10	.0179	−1.10	.1337	−.10	.4502	.90	.8159	1.90	.9713	2.90	.9931
−2.05	.0202	−1.05	.1469	−.05	.4301	.95	.8289	1.95	.9744	2.95	.9984

or:

$$N(d_2) = .7713$$

Next, these values are plugged into the Black–Scholes formula to get the call option's price as follows:

$$\begin{aligned} C &= N(d_1)S - E(e^{-rT})N(d_2) \\ &= .7891(60) - 58(e^{-.05(3/12)}).7713 \\ &= 47.3439 - 44.1797 = 3.164 \end{aligned}$$

The Black–Scholes model can also be used to price European put options. The put option pricing model is presented in the following equation:

$$P = -N(-d_1)S + E(e^{-rT})N(-d_2)$$

where

P = put option price

All other variables are the same as above.

Example 10–4 Using the Black–Scholes Formula to Value a Put Option

Suppose you own a put option on the stock described in Example 10–5. The put option has an exercise price of \$65. The values of $d_1 = -.8034$ and of $d_2 - .7434$. The values of $N(d_1)$ and $N(d_2)$ are found from Table 10–11, which shows the cumulative normal distribution. Interpolation from Table 10–11 gives $N(d_1)$:

d_1	$N(d_1)$
−.85	.1977
−.8034	?
−.80	.2119

or:

$$N(d_1) = .2109$$

and $N(d_2)$:

d_2	$N(d_2)$
$-.75$	.2266
$-.7434$	?
$-.70$	.2420

or:

$$N(d_2) = .2286$$

Next, these values are plugged into the Black–Scholes formula to get the call option's price as follows:

$$\begin{aligned} P &= -N(-d_1)S + E(e^{-rT})N(-d_2) \\ &= -.2109(60) + 65(e^{-.05(3/12)}).2286 \\ &= -12.654 + 14.674 = 2.020 \end{aligned}$$