Chapter 8

1. Let X be the total cost. The equation of value is

$$X = \left(\frac{X}{10}\right)\ddot{a}_{\overline{12}|j} \quad \text{where } j \text{ is the monthly rate of interest or } \ddot{a}_{\overline{12}|j} = 10.$$

The unknown rate j can be found on a financial calculator as 3.503%. The effective rate of interest i is then

$$i = (1+j)^{12} - 1 = (1.03503)^{12} - 1 = .512, \text{ or } 51.2\%.$$

2. Per dollar of loan we have

$$L = 1 \quad K = .12 \quad n = 18 \quad R = 1.12/18$$

and the equation of value

$$\frac{1.12}{18} a_{\overline{18}|j} = 1 \quad \text{or} \quad a_{\overline{18}|j} = 16.07143.$$

The unknown rate j can be found on a financial calculator as .01221. The APR is then

$$\text{APR} = 12j = (12)(.01221) = .1465, \text{ or } 14.65\%.$$

3. The equation of value is

$$7.66 a_{\overline{16}|j} = 100 \quad \text{or} \quad a_{\overline{16}|} = 13.05483.$$

The unknown rate j can be found on a financial calculator as .025. The APY is then

$$\text{APY} = (1+j)^{12} - 1 = (1.025)^{12} - 1 = .3449, \text{ or } 34.49\%.$$

4. (*a*) Amount of interest = Total payments − Loan amount

Option A: $13(1000) - 12,000 = 1,000.00.$

Option B: $12 \cdot \left(\dfrac{12,000}{a_{\overline{12}|.01}}\right) - 12,000 = 794.28.$

Difference in the amount of interest = 1,000.00 − 794.28 = $205.72.

(*b*) The equation of value is

$$12,000 - 1000 = 1000 a_{\overline{12}|j} \quad \text{or} \quad a_{\overline{12}|j} = 11.$$

Using a financial calculator $j = .013647$ and $\text{APR} = 12j = .1638$, or 16.38%.

(*c*) $\text{APR} = 12(.01) = 12.00\%,$ since the amortization rate directly gives the APR in the absence of any other fees or charges.

5. Bank 1: $X = \dfrac{L + (2)(.065)L}{24} = .04708L.$

Bank 2: We have $j = (1.126)^{1/12} - 1 = .00994$ so that $Y = \dfrac{L}{a_{\overline{24|}j}} = .04704L.$

Bank 3: We have $j = .01$ and $Z = \dfrac{L}{a_{\overline{24|}j}} = .04707L.$

Therefore $Y < Z < X$.

6. (*a*) The United States Rule involves irregular compounding in this situation. We have

$$B_3 = 8000 - [2000 - (8000)(.03)] = 6240$$
$$B_9 = 6240 - [4000 - (6240)(.06)] = 2614.40$$
$$X = B_{12} = 2614.40(1.03) = \$2692.83.$$

(*b*) The Merchant's Rule involves simple interest throughout. We have

$$X = 8000(1.12) - 2000(1.09) - 4000(1.03)$$
$$= \$2660.00.$$

7. (*a*) The interest due at time $t = 1$ is $10{,}000(.1) = 1000$. Since only 500 is paid, the other 500 is capitalized. Thus, the amount needed to repay the loan at time $t = 2$ is $10{,}500(1.1) = \$11{,}550.$

(*b*) Under the United States Rule, the interest is still owed, but is not capitalized. Thus, at time $t = 2$ the borrower owes 500 carryover from year 1, 1000 in interest in year 2, and the loan repayment of 10,000 for a total of $11,500.

8. (*a*) The equation of value is

$$200(1+i)^2 - 1000(1+i) + 1000 = 0$$
$$(1+i)^2 - 5(1+i) + 5 = 0.$$

Now solving the quadratic we obtain

$$1 + i = \frac{5 \pm \sqrt{(-5)^2 - (4)(1)(5)}}{(2)(1)} = \frac{5 \pm \sqrt{5}}{2}$$
$$= 1.382 \text{ and } 2.618$$

so that $i = .382$ and 2.618, or 38.2% and 261.8%.

(*b*) The method of equated time on the payments is

$$\bar{t} = \frac{(200)(0)+(1000)(2)}{1200} = \frac{5}{3}.$$

This method then uses a loan of 1000 made at time $t=1$ repaid with 1200 at time $t = 5/3$. The equation of value is $1000(1+j)=1200$ or $j=.20$ for 2/3 of a year. Thus, the $\text{APR} = 3/2\,j = .30,$ or 30%.

9. Consider a loan $L = a_{\overline{n}|}$ with level payments to be repaid by *n* payments of 1 at regular intervals. Instead the loan is repaid by *A* payments of 1 each at irregular intervals. Thus, $A - a_{\overline{n}|}$ represents the finance charge, i.e. total payments less the amount of loan.

   If *B* is the exact single payment point then $A(1+i)^{-B}$ is the present value of total payments or the amount of the loan. Thus, $A - A(1+i)^{-B}$ is again the finance charge.

   $C/1000$ is the finance charge per 1000 of payment and there are *A* payments. Thus, $C\left(\frac{A}{1000}\right)$ is the total finance charge.

10. The monthly payments are:

$$\text{Option A} = \frac{10{,}000}{a_{\overline{48}|.10/12}} = 253.626.$$

$$\text{Option B} = \frac{9000}{a_{\overline{48}|.09/12}} = 223.965.$$

   To make the two options equal we have the equation of value

$$(253.626 - 223.965)\,s_{\overline{48}|.09/12} = 1000(1+i)^4$$

   and solving for the effective rate *i*, we obtain $i = .143,$ or 14.3%.

11. (*a*) The monthly payments are

$$\text{Option A} = \frac{16{,}000}{24} = 666.67.$$

$$\text{Option B} = \frac{15{,}500}{a_{\overline{24}|.0349/12}} = 669.57.$$

   Option A has the lower payment and thus is more attractive.

(*b*) If the down payment is *D*, then the two payments will be equal if

$$\frac{16,000-D}{24}=\frac{15,500-D}{a_{\overline{24}|.0349/12}}.$$

Therefore,

$$D=\frac{(15,500)(24)-16,000a_{\overline{24}|.0349/12}}{24-a_{\overline{24}|.0349/12}}$$

$$=\$1898 \text{ to the nearest dollar.}$$

12. The monthly rate of interest equivalent to 5% effective is $j=(1.05)^{1/12}-1=.004074$. Thus, the monthly loan payment is

$$R=\frac{15,000}{a_{\overline{48}|.004074}}=344.69.$$

The present value of these payments at 12% compounded monthly is

$$344.69a_{\overline{48}|.01}=13,089.24.$$

Thus, the cost to the dealer of the inducement is

$$15,000-13,089.24=\$1911 \text{ to the nearest dollar.}$$

13. (*a*) Prospective loan balance for A is

$$\frac{20,000}{a_{\overline{48}|.07/12}}a_{\overline{36}|.07/12}=\$15,511>\$15,000.$$

Prospective loan balance for B is

$$\frac{20,000}{a_{\overline{24}|.07/12}}a_{\overline{12}|.07/12}=\$10,349<\$15,000.$$

(*b*) The present value of the cost is the present value of the payments minus the present value of the equity in the automobile.

Cost to A:

$$\frac{20,000}{a_{\overline{48}|.07/12}}a_{\overline{12}|.005}-(15,000-15,511)(1.005)^{-12}=(478.92)(11.62)+(511)(.942)$$

$$=\$6047 \text{ to the nearest dollar.}$$

Cost to B:

$$\frac{20,000}{a_{\overline{24}|.07/12}}a_{\overline{12}|.005}-(15,000-10,349)(1.005)^{-12}=(895.45)(11.62)-(4651)(.942)$$

$$=\$6026 \text{ to the nearest dollar.}$$

14. (*a*) Formula (8.6) is

$$R = B_0 i + \frac{D}{s_{\overline{n}|}}$$
$$= (20,000)(.005) + \frac{20,000 - 13,000}{s_{\overline{24}|.005}}$$
$$= \$375.24.$$

(*b*) The equation of value is

$$20,000 - 300 = 375.24 a_{\overline{24}|j} + (13,000 + 200) v_j^{24}$$
$$19,700 = 375.24 a_{\overline{24}|j} + 13,200 v_j^{24}.$$

Using a financial calculator, we find that $j = .63\%$ monthly.

(*c*) The equation of value is

$$20,000 - 300 = 375.24 a_{\overline{12}|j} + (16,000 + 800) v_j^{12}$$
$$19,700 = 375.24 a_{\overline{12}|j} + 16,800 v_j^{12}.$$

Using a financial calculator, we find that $j = .73\%$ monthly.

15. We modify the formula in Example 8.4 part (2) to

$$19,600 - 341.51 = 341.51 a_{\overline{36}|j} + (10,750 + 341.51) v_j^{36}$$
$$19,258.49 = 341.51 a_{\overline{36}|j} + 11,091.51 v_j^{36}.$$

Using a financial calculator, we find that $j = .74\%$ monthly. The nominal rate of interest convertible monthly is $12j = 8.89\%$. This compares with the answer of 7.43% in Example 8.4. Thus, the effect of making a security deposit that does not earn interest is significant.

16. (*a*) The NPV of the "buy" option is

$$50,000(1.01)^{-72} - \left(400,000 + 4000 a_{\overline{72}|.01}\right) = 24,424.80 - 604,601.57 = -\$580,177.$$

(*b*) The NPV of the "lease" option is

$$-12,000 a_{\overline{72}|.01} = -\$613,805.$$

(*c*) The "buy" option should be chosen since it is the least negative.

17. (*a*)

Mortgage loan L	=	$.75(160,000)=120,000.$
Mortgage payment R	=	$\dfrac{120,000}{a_{\overline{360}\rvert .0075}}=965.55.$
Interest as points Q	=	$(.015)(120,000)=1800.00.$
September 16 interest	=	$.09\left(\dfrac{15}{365}\right)(120,000)=443.84.$
November 1 interest	=	$(.0075)(120,000)=900.00.$
December 1 interest	=	$.0075[120,000-(965.55-900.00)]$
	=	$899.51.$
Total interest	=	$1800.00+443.84+900.00+899.51$
	=	$\$4043.35.$

(*b*)

Interest as points	Q	=	$(.015)(120,000)=1800.00$
Adjusted loan	L^*	=	$120,000-1800=118,200$
APR calculation $a_{\overline{360}\rvert j}$ =	$\dfrac{L^*}{R}$	=	$\dfrac{118,200}{965.55}=122.41728.$

Use a financial calculator to find $j=.007642$, $\text{APR}=12j=12(.007642)=.0917$, or 9.17%.

18. The interest saved by this payment scheme is the interest in each even-numbered payment in the original $12\times 15=180$ payment amortization schedule. Thus, we have

$$\begin{aligned}
&1000\left[(1-v^{179})+(1-v^{177})+\cdots+(1-v)\right]\\
&=1000\left[90-(v+v^3+\cdots+v^{177}+v^{179})\right]\\
&=90,000-1000v(1+v^2+\cdots+v^{176}+v^{178})\\
&=90,000-1000v\frac{1-v^{180}}{1-v^2}\\
&=90,000-1000(1+i)\left[\frac{1-v^{180}}{(1+i)^2-1}\right]\\
&=90,000-1000\frac{\ddot{a}_{\overline{180}\rvert}}{s_{\overline{2}\rvert}}.
\end{aligned}$$

19. At time $t=2$ the accumulated value of the construction loan is

$$1,000,000(1.075)^4+500,000(1.075)^3+500,000(1.075)^2=2,534,430.08$$

which becomes the present value of the mortgage payments. Thus, we have the equation of value

$$2,534,430.08 = P\ddot{a}_{\overline{60}|.01} + 2P\ddot{a}_{\overline{300}|.01}(1.01)^{-60}$$

$$\text{and} \quad P = \frac{2,534,430.08}{\ddot{a}_{\overline{60}|.01} + 2(1.01)^{-60}\ddot{a}_{\overline{300}|.01}} = \$16,787$$

to the nearest dollar. The 12th mortgage payment is equal to *P*, since it is before the payment doubles. Also, note the annuity-due, since the first mortgage payment is due exactly two years after the initial construction loan disbursement.

20. The loan origination fee is $.02(100,000) = 2000.$

The mortgage payment is $R = \dfrac{100,000}{a_{\overline{30}|.08}} = 8882.74.$

Loan balance at $t = 1$: $B_1 = 100,000(1.08) - 8882.74 = 99,117.26.$

Loan balance at $t = 2$: before any payments $B_2' = 99,117.26(1.08) = 107,046.64.$

Adjusted loan $L^* = 100,000 - 2000 = 98,000.$

Thus, the equation of value becomes

$$98,000 = 8882.74v + 107,046.64v^2$$

and solving the quadratic

$$v = \frac{-8882.74 \pm \sqrt{(8882.74)^2 - (4)(107,046.64)(-98,000)}}{2(107,046.64)}$$

$$= .91622 \text{ rejecting the negative root.}$$

Finally, $i = \dfrac{1}{v} - 1 = .0914,$ or 9.14%.

21. There are $10 \times 4 = 40$ payments on this loan. The quarterly interest rates are $j_1 = \dfrac{.12}{4} = .03$ and $\dfrac{j_2}{4} = .035.$ The loan balance $B_{12} = 1000a_{\overline{28}|.03} = 18,764.12.$ The loan balance after 12 more payments is

$$B_{24} = (18,764.12)(1.035)^{12} - 1000s_{\overline{12}|.035}$$

$$= \$13,752 \text{ to the nearest dollar.}$$

22. (*a*) The equation of value is

$$100,000 = R\left[v + 1.05v^2 + (1.05)^2 v^3 + (1.05)^3 v^4 + (1.05)^4 v^5 + (1.05)^4 v^6 + (1.05)^4 v^7 + \cdots + (1.05)^4 v^{30}\right]$$

$$= \frac{R}{1.09}\left[1 + \frac{1.05}{1.09} + \left(\frac{1.05}{1.09}\right)^2 + \left(\frac{1.05}{1.09}\right)^3 + \left(\frac{1.05}{1.09}\right)^4 \left(1 + v + \cdots + v^{25}\right)\right]$$

and $R = \dfrac{100,000(1.09)}{\dfrac{1-\left(\frac{1.05}{1.09}\right)^5}{1-\left(\frac{1.05}{1.09}\right)} + \left(\frac{1.05}{1.09}\right)^4 a_{\overline{25}|.09}} = \$8318$ to the nearest dollar.

(*b*) $I_1 = (.09)(100,000) = 9000$ and

$R_1 = \$8318;$ so, yes, negative amortization does occur.

23. The payment on the assumed mortgage is

$$R_1 = \frac{60,000}{a_{\overline{30}|.08}} = 5329.64.$$

The loan balance $B_{10} = 5329.64 a_{\overline{20}|.08} = 52,327.23$. The amount of the "wraparound" mortgage is $(.85)(120,000) - 52,327.23 = 49,672.77$. The payment on the "wraparound" mortgage is $R_2 = \dfrac{49,672.77}{a_{\overline{20}|.10}} = 5834.54$. The total payment required is $R_1 + R_2 = \$11,164$ to the nearest dollar.

24. The equity in the house will be

$$100,000(1.06)^5 - 500 s_{\overline{60}|.01} = 133,882.56 - 40,834.83 = \$92,988$$

to the nearest dollar.

25. The monthly payment is $\dfrac{1200+108}{12} = 109.$

(*a*) All the early payments are principal, so

$$B_4 = 1200 - 4(109) = \$764.$$

(*b*) All interest is paid from the first payment, so

$$B_4 = 1200 - (109 - 108) - 3(109) = \$872.$$

(*c*) The ratio 1200/1308 of each payment is principal, so

$$B_4 = 1200 - 4\left(\frac{1200}{1308}\right)(109) = \$800.$$

(*d*) The interest in the first four payments is

$$\left(\frac{12+11+10+9}{78}\right)(108) = 58.15, \text{ so}$$
$$B_4 = 1200 - 4(109) + 58.15 = \$822.15.$$

26. Under the direct ratio method

$$I_2 = K\frac{8}{S_9} = 20 \quad \text{and} \quad I_8 = K \cdot \frac{2}{S_9}.$$

Therefore $I_8 = \frac{2}{8}(20) = \$5.$

27. The total payments are $6(50) + 6(75) = 750$. Now, $K = 750 - 690 = 60$, so that $60/750 = .08$ of each payment is interest and .92 is principal. Therefore, principal payments are 46 for the first six months and 69 for the last 6 months. The 12 successive loan balances are:

690, 644, 598, 552, 506, 460, 414, 345, 276, 207, 138, 69

which sum to 4899. We then have

$$i^{cr} = \frac{(12)(60)}{4899} = .147, \text{ or } 14.7\%.$$

28. We are given:

$$i^{max} = \frac{2mK}{L(n+1) - K(n-1)} = .20 \quad \text{and} \quad i^{min} = \frac{2mK}{L(n+1) + K(n-1)} = .125.$$

Taking reciprocals

$$\frac{L(n+1)}{2mK} - \frac{K(n-1)}{2mK} = 5 \quad \text{and} \quad \frac{L(n+1)}{2mK} + \frac{K(n-1)}{2mK} = 8.$$

We have two equations in two unknowns which can be solved to give

$$\frac{L(n+1)}{2mK} = 6.5 \quad \text{and} \quad \frac{K(n-1)}{2mK} = 1.5.$$

Now taking the reciprocal of formula (8.19)

$$\frac{1}{i^{dr}} = \frac{L(n+1) + \frac{1}{3}K(n-1)}{2mK} = 6.5 + \tfrac{1}{3}(1.5) = 7$$

so that $i^{dr} = \frac{1}{7} = .143,$ or 14.3%.

29. For annual installments R, we have $m = 1$ and $n = 5$. The finance charge is $K = 5R - L$. We then have

$$B_2^{dr} = 3P - \frac{6}{15}(5R - L) = R + \frac{6}{15}L.$$

For the amortized loan, we have

$$B_2^P = Ra_{\overline{3}|.05} = 2.72317R.$$

Equating the two we have

$$\frac{6}{15}L = 1.72317R \quad \text{or} \quad L = 4.30793R.$$

However, since $L = Ra_{\overline{5}|\,i}$, we have $a_{\overline{5}|\,i} = 4.31.$

30. We have

$$I_1 = \frac{i}{m} \cdot L$$

$$I_2 = \frac{i}{m}\left[L - \frac{L+K}{n}\right]$$

$$\vdots$$

$$I_n = \frac{i}{m}\left[1 - (n-1)\left(\frac{L+K}{n}\right)\right].$$

However, these interest payments do not earn additional interest under simple interest. The finance charge is the sum of these interest payments

$$K = \sum_{t-0}^{n-1} \frac{i}{m}\left[1 - t \cdot \frac{L+K}{n}\right] = \frac{i}{m}\left[Ln - \frac{L+K}{n} \cdot \frac{n(n-1)}{2}\right]$$

which can be solved to give formula (8.14)

$$i^{max} = \frac{2mK}{L(n+1) - K(n-1)}.$$

31. The reciprocal of the harmonic mean is the arithmetic mean of the two values given. In symbols,

$$\frac{1}{2}\left[\frac{1}{i^{max}} + \frac{1}{i^{min}}\right] = \frac{1}{2}\left[\frac{L(n+1) - K(n-1)}{2mK} + \frac{L(n+1) + K(n-1)}{2mK}\right]$$
$$= \frac{1}{2}\left[\frac{2L(n+1)}{2mK}\right] = \frac{L(n+1)}{2mK} = \frac{1}{i^{cr}}.$$

32. (*a*) The outstanding loan balances are

$$L, L - \left(\frac{L+K}{n}\right), L - 2\left(\frac{L+K}{n}\right), \ldots, L - (n-r)\left(\frac{L+K}{n}\right)$$

after $n - r$ payments have been made. Since *r* payments are enough to pay *K*, then $B_{n-r+1} = 0$. The denominator of formula (8.13) then becomes

$$(n-r+1)L - \left(\frac{L+K}{n}\right)\left[\frac{(n-r)(n-r+1)}{2}\right].$$

Finally, applying formula (8.13) and multiplying numerator and denominator by $2n$, we obtain

$$i^{max} = \frac{2mnK}{2n(n-r+1)L - (n-r)(n-r+1)(L+K)}.$$

(*b*) The first $r-1$ payments are all interest, so that the outstanding balances are all equal to *L* followed by

$$(n-r)\left(\frac{L+K}{n}\right), (n-r+1)\left(\frac{L+K}{n}\right), \ldots, \frac{L+K}{n}.$$

Again applying formula (8.13)

$$i = \frac{mK}{rL + r\left(\frac{L+K}{n}\right)\left[\frac{(n-r)(n-r+1)}{2}\right]} = \frac{2mnK}{2nrL + (n-r)(n-r+1)(L+K)}.$$

33. (*a*) (1) $D_3 = \dfrac{A}{s_{\overline{10}|}}(1+j)^2$ and $D_9 = \dfrac{A}{s_{\overline{10}|}}(1+j)^8$

Therefore, $D_9 = D_3(1+j)^6 = (1000)(1.05)^6 = \$1340.10$.

(2) $D_9 = D_3 = \$1000.00$.

(3) $D_3 = \dfrac{8A}{S_{10}}$ and $D_9 = \dfrac{2A}{S_{10}}$.

Therefore, $D_9 = \dfrac{1}{3}D_3 = \dfrac{1}{4}(1000) = \$250.00.$

(*b*) (1) $D_3 = \dfrac{A}{s_{\overline{10}|}}(1.05)^2 = 1000,$ so that $A = 1000 s_{\overline{10}|}\, v^2 = \$11,408.50.$

(2) $D_3 = \dfrac{A}{10} = 1000,$ so that $A = \$10,000.00$

(3) $D_3 = \dfrac{8A}{S_{10}} = \dfrac{8A}{\frac{1}{2}(10)(11)} = 1000,$ so that $A = \dfrac{(1000)(10)(11)}{(2)(8)} = \$6875.00$.

34. The present value of the depreciation charges is

$$\sum_{t=1}^{10} \frac{2000-400}{s_{\overline{10}|\,i}}(1+i)^{t-1} v_i^t = \sum_{t=1}^{10} \frac{1600}{s_{\overline{10}|\,i}(1+i)} = \frac{16,000}{\ddot{s}_{\overline{10}|\,i}} = 1000, \text{ or } \ddot{s}_{\overline{10}|\,i} = 16.$$

Using a financial calculator, we obtain $i = .0839,$ or 8.39%.

35. We have the following:

(*i*) $D = \dfrac{X-Y}{n} = 1000$ or $X - Y = 1000n.$

(*ii*) $D_3 = \dfrac{n-3+1}{S_n}(X-Y) = \dfrac{n-2}{\frac{1}{2}n(n+1)}(X-Y) = 800$

or $(n-2)(X-Y) = 400n(n+1).$

Now substituting (*i*) into (*ii*), we have

$$1000n(n-2) = 400n(n+1)$$
$$1000n - 2000 = 400n + 400$$
$$600n = 2400 \text{ or } n = 4.$$

Therefore, $X - Y = 4000.$

$$(iii)\ d = 1 - \left(\frac{Y}{X}\right)^{.25} = .33125 \quad \text{or} \quad \left(\frac{Y}{X}\right)^{.25} = .66875$$

$$\frac{Y}{X} = (.66875)^4 = .2 \qquad\qquad Y = .2X.$$

Therefore, $X - .2X = 4000$, and $X = \$5000.$

36. Under the constant percentage method

$$\begin{array}{lclcl} D_1 & = & .2B_0 = .2(20{,}000) & = & 4000 \\ D_2 & = & .2B_1 = .2(16{,}000) & = & 4000(.8) \\ D_3 & = & .2B_2 = .2(12{,}800) & = & 4000(.8)^2 \\ \vdots & & & & \vdots \\ D_{15} & & & = & 4000(.8)^{14} \end{array}$$

The depreciation charges constitute an annuity whose payments vary in geometric progression. The accumulated value is

$$4000\left[(1.06)^{14} + (.8)(1.06)^{13} + \cdots + (.8)^{13}(1.06) + (.8)^{14}\right]$$

$$= 4000\frac{\left[\frac{(1.06)^{15}}{.8} - (.8)^{14}\right]}{\frac{1.06}{.8} - 1} = \$36{,}329 \text{ to the nearest dollar.}$$

37. Under the sum of the years digits method

$$(5000 - S)\frac{10+9+8+7}{55} = 5000 - 2218 = 2782$$

and solving $S = 5000.$ The level depreciation charge over the next six years will be

$$\frac{2218 - 500}{6} = \$286.33.$$

38. Machine I: $B_{18} = S + \frac{S_2}{S_{20}}(A - S) = 5000 + \frac{3}{210}(35{,}000) = 5500.$

Machine II: $B_{18} = A - \frac{A-S}{s_{\overline{20}|}} s_{\overline{18}|} = 5346.59 + .86633S.$

Equating the two and solving for S gives

$$S = \frac{5500 - 5346.59}{.86633} = \$177 \text{ to the nearest dollar.}$$

39. Under the compound interest method

$$B_{10} = A - \left(\frac{A-S}{s_{\overline{15}|}}\right)s_{\overline{10}|} = 15,000 - \left(\frac{13,000}{21.5786}\right)(12.5778) = 7422.52.$$

Continuing thereafter on the straight-line method gives

$$B_{12} = 7422.52 - \frac{2}{5}(7442.52 - 2000) = \$5253 \text{ to the nearest dollar.}$$

40. Machine A: $D = \dfrac{2450 - 1050}{14} = 100$

and the present value of these depreciation charges is

$$100a_{\overline{14}|.10} = 736.67.$$

Machine B: $S_{14} = \dfrac{1}{2}(14)(15) = 105.$

The pattern of depreciation charges is

$$\frac{14}{105}(Y-1050), \frac{13}{105}(Y-1050), \ldots, \frac{1}{105}(Y-1050).$$

The present value of these depreciation charges is

$$\frac{Y-1050}{105}\left(14v^{14} + 13v^{13} + \cdots + v^{14}\right) = \frac{Y-1050}{105}(Da)_{\overline{14}|}.$$

Now evaluating $(Da)_{\overline{14}|} = \dfrac{14 - a_{\overline{14}|.1}}{.1} = 66.3331$

we obtain

$$\frac{(Y-1050)(66.3331)}{105} = 736.67$$

and solving $Y = \$2216$ to the nearest dollar.

41. We have

$$\frac{d}{dt}\left(B_t^{SL} - B_t^{CP}\right) = \frac{d}{dt}\left[\left\{A - \frac{t}{n}(A-S)\right\} - A(1-d)^t\right]$$

$$= \frac{-A-S}{n} - A(1-d)^t \ln(1-d) = 0.$$

Now $A(1-d)^n = S$, so that $1-d = (S/A)^{1/n}$. Substituting for $1-d$, we obtain

$$\frac{A-S}{n}=-A\left[(S/A)^{1/n}\right]^t\ln\left[(S/A)^{1/n}\right].$$

After several steps of algebraic manipulation we find that

$$t=n\frac{\ln(1-S/A)-\ln\left[-\ln(S/A)\right]}{n\ln(S/A)}.$$

42. (*a*) $H=10{,}000(.05)+\dfrac{9000}{s_{\overline{10|}.05}}+500=\$1715.55.$

(*b*) $K=\dfrac{1715.55}{.05}=\$34{,}311$ to the nearest dollar.

43. Equating periodic charges, we have

$$1000i+\frac{950}{s_{\overline{9|}}}=1100i+\frac{900}{s_{\overline{9|}}}.$$

This simplifies to

$$\frac{50}{s_{\overline{9|}}}=100i \qquad \text{or} \qquad 50=100\left[(1+i)^9-1\right]$$

$$(1+i)^9=1.5 \quad \text{and} \quad i=(1.5)^{1/9}-1=.0461, \text{ or } 4.61\%.$$

44. Plastic trays:

To cover 48 years, six purchases will be necessary at the prices:

$$20, 20(1.05)^8, 20(1.05)^{16}, 20(1.05)^{24}, 20(1.05)^{32}, 20(1.05)^{40}.$$

The present value of these purchases is

$$20\left[1+\left(\frac{1.05}{1.1025}\right)^8+\left(\frac{1.05}{1.1025}\right)^{16}+\cdots+\left(\frac{1.05}{1.1025}\right)^{40}\right]$$

$$=20\left[1+(1.05)^{-8}+(1.05)^{-16}+\cdots+(1.05)^{-40}\right]$$

$$=20\frac{1-v^{48}}{1-v^8}=55.939.$$

Metal trays:

Two purchases will be necessary at the prices: $X, X(1.05)^{24}$.

The present value of these purchases is

$$X\frac{1-v^{48}}{1-v^{24}}=1.3101X.$$

Therefore, $1.3101X=55.939$ or $X=\$42.70$.

45. Without preservatives the periodic charge for the first 14 years is

$$H = 100i + \frac{100}{s_{\overline{14}|}} = \frac{100}{a_{\overline{14}|}}.$$

For the next 14 years it is $H(1.02)^{14}$, continuing indefinitely. Thus, the capitalized cost is

$$K = Ha_{\overline{14}|} + H(1.02)^{14} v^{14} a_{\overline{14}|} + \cdots$$
$$= Ha_{\overline{14}|}\left[1 + \left(\frac{1.02}{1.04}\right)^{14} + \cdots\right] = 100\left[\frac{1}{1 - \left(\frac{1.02}{1.04}\right)^{14}}\right] = 420.108.$$

With preservatives we replace 100 with $100 + X$ and 14 with 22 to obtain

$$K = (100 + X)\left[\frac{1}{1 - \left(\frac{1.02}{1.04}\right)^{22}}\right] = 2.87633(100 + X).$$

Equating and solving for X we obtain

$$X = \frac{420.108}{2.87633} - 100 = \$46.06.$$

46. We can equate periodic charges to obtain

$$1000(.035) + \frac{950}{s_{\overline{10}|.035}} = (1000 + X)(.035) + \frac{950 + X}{s_{\overline{15}|.035}}$$
$$\frac{950}{s_{\overline{10}|.035}} = X(.035) + \frac{950 + X}{s_{\overline{15}|.035}}$$
$$80.9793 = .035X + 49.2338 + .05183X$$
$$\text{and } X = \frac{31.7455}{.08683} = \$365.63.$$

47. Machine 1:

For the first 20 years periodic charges are

$$H_1 = 100{,}000i + \frac{100{,}000}{s_{\overline{20}|}} + 3000(1.04)^{t-1} = \frac{100{,}000}{a_{\overline{20}|}} + 3000(1.04)^{t-1}$$
$$\text{for } t = 1, 2, \ldots, 20.$$

The present value is

$$100{,}000 + 3000\left[1 + \left(\frac{1.04}{1.08}\right) + \left(\frac{1.04}{1.08}\right)^2 + \cdots + \left(\frac{1.04}{1.08}\right)^{19}\right] = 142{,}921.73.$$

For the next 20 years it is $H(1.04)^{20}$ continuing indefinitely. Thus, the capitalized cost is

$$142,921.73\left[1+\left(\frac{1.04}{1.08}\right)^{20}+\left(\frac{1.04}{1.08}\right)^{40}+\cdots\right]=269,715.55.$$

Machine 2:

$$H_2=\frac{A}{a_{\overline{15|}}}+10,000(1.04)^{t-1} \quad \text{for} \quad t=1,2,\ldots,15.$$

The present value is

$$X+10,000\left[1+\left(\frac{1.04}{1.08}\right)+\cdots+\left(\frac{1.04}{1.08}\right)^{14}\right]=116,712.08+X.$$

The capitalized cost is

$$(116,712.08+A)\left[1+\left(\frac{1.04}{1.08}\right)^{15}+\left(\frac{1.04}{1.08}\right)^{30}+\cdots\right]=(2.31339)(116,712.08+A).$$

Since Machine 2 produces output twice as fast as Machine 1, we must divide by 2 before equating to Machine 1. Finally, putting it all together we obtain

$$A=\frac{2(269,715.55)}{2.31339}-116,712.08=\$116,500 \text{ to the nearest } \$100.$$

48. The sinking fund deposit is

$$D=\frac{A-S}{s_{\overline{n|}j}}.$$

From (*i*), (*ii*), and (*iii*) we obtain

$$B_6=A-Ds_{\overline{6|}.09} \quad \text{or} \quad 55,216.36=A-7.52334D.$$

From (*ii*), (*v*), and (*vi*) we obtain

$$H=Ai+\frac{A-S}{s_{\overline{n|}j}}+M \quad \text{or}$$

$$11,749.22=.09A+D+3000.$$

Thus, we have two equations in two unknowns which can be solved to give

$$D=2253.74 \quad \text{and} \quad A=\$72,172.$$