Chapter 10

1. (*a*) We have

$$1000\left[(1.095)^{-1}+(1.0925)^{-2}+(1.0875)^{-3}+(1.08)^{-4}+(1.07)^{-5}\right]=\$3976.61.$$

(*b*) The present value is greater than in Example 10.1 (1), since the lower spot rates apply over longer periods while the higher spot rates apply over shorter periods.

2. We have

$$1000\left[1+(1.05)(1+s_1)^{-1}+(1.05)^2(1+s_2)^{-2}+(1.05)^3(1+s_3)^{-3}+(1.05)^4(1+s_4)^{-4}\right]$$
$$=1000\left[1+\frac{1.05}{1.09}+\left(\frac{1.05}{1.081}\right)^2+\left(\frac{1.05}{1.0729}\right)^3+\left(\frac{1.05}{1.06561}\right)^4\right]=\$4786.78.$$

3. Since s_k is differentiable over $0 \le k \le 4$,

$$\frac{d}{dk}s_k=.002-.001k=0 \text{ at } k=2$$

which is a relative maximum or minimum. Computing values for $k=0,1,2,3,4$ we obtain

$$s_0=.09 \quad s_1=.0915 \quad s_2=.092 \quad s_3=.0915 \quad s_4=.09.$$

(*a*) Normal.

(*b*) Inverted.

4.

Payment at	Spot rate	Accumulated value
$t=0$	.095	$(1.095)^5=1.57424$
$t=1$	.0925 − .0025	$(1.09)^4=1.41158$
$t=2$	.0875 − .0050	$(1.0825)^3=1.26848$
$t=3$	.0800 − .0075	$(1.0725)^2=1.15026$
$t=4$	.0700 − .0100	$(1.06)^1=1.06000$
		6.4646

5. Adapting Section 9.4 to fit this situation we have

$$\frac{10,000(1.0925)^4}{(1.05)^2(1.04)^2} = \$11,946.50.$$

6. (*a*) $2P_B - P_A = 2(930.49) - 1019.31 = \$841.67.$

   (*b*) $2C_B - C_A = 2(1000.00) - 1000.00 = \$1000.00.$

   (*c*) We have $s_2 = .09$ and $841.67(1.09)^2 = 1000.00$ confirming the statement.

7. The price of the 6% bond per 100 is

$$P_{.06} = 6a_{\overline{6}|.12} + 100(1.12)^{-6} = 75.33.$$

   The price of the 10% bond per 100 is

$$P_{.10} = 10a_{\overline{6}|.08} + 100(1.08)^{-6} = 109.25.$$

   We can adapt the technique used above in Exercise 6. If we buy 10/6 of the 6% bonds, the coupons will exactly match those of the 10% bond. The cost will be

$$\frac{10}{6}(75.33) = 125.55 \text{ and will mature for } \frac{10}{6}(100).$$

   Thus, we have

$$(125.55 - 109.25)(1 + s_6)^6 = \frac{4}{6}(100)$$

   and solving $s_6 = .2645,$ or 26.45%.

8. Applying formula (10.4)

   (*a*) $\dfrac{1-(1.08)^{-2}}{(1.07)^{-1}+(1.08)^{-2}} = .0796,$ or $7.96\%.$

   (*b*) $\dfrac{1-(1.09)^{-3}}{(1.07)^{-1}+(1.08)^{-2}+(1.09)^{-3}} = .0888,$ or $8.88\%.$

   (*c*) The yield curve has a positive slope, so that the at-par yield rate increases with *t*.

9. (*a*) Since $6\% < 8.88\%,$ it is a discount bond.

   (*b*) $P = 60\left[(1.07)^{-1} + (1.08)^{-2} + (1.09)^{-3}\right] + 1000(1.09)^{-3} = 926.03.$

   The amount of discount is $1000.00 - 926.03 = \$73.97.$

10. (*a*) We have

$$1030 = 100\sum_{t=1}^{3}(1+s_t)^{-t} + 1000(1+s_3)^{-3}$$
$$= 100\sum_{t=1}^{3}(1+s_t)^{-t} + 1000(1.08)^{-3}$$

and

$$100\sum_{t=1}^{3}(1+s_t)^{-t} = 1030 - 793.832 = 236.168.$$

Then

$$1035 = 100\sum_{t=1}^{4}(1+s_t)^{-t} + 1000(1+s_4)^{-4}$$
$$= 236.168 + 1100(1+s_4)^{-4}$$

and solving we obtain

$$s_4 = .0833, \text{ or } 8.33\%.$$

(*b*) We have

$$1037 = 100\sum_{t=1}^{5}(1+s_t)^{-t} + 1000(1+s_5)^{-5}$$
$$= 236.168 + 100(1.0833)^{-4} + 1100(1+s_5)^{-5}$$

and solving we obtain

$$s_5 = .0860, \text{ or } 8.60\%.$$

(*c*) $P = 100\sum_{t=1}^{6}(1+s_t)^{-t} + 1000(1+s_6)^{-6}$

$$= 1037 - 100(1.0860)^{-5} + 1100(1.07)^{-6} = \$1107.99.$$

11. (*a*) We have

$$P = 60\left[(1.07)^{-1} + (1.08)^{-2}\right] + 1060(1.09)^{-3} = \$926.03.$$

(*b*) Use a financial calculator setting

$$\text{N} = 3 \quad \text{PV} = -926.03 \quad \text{PMT} = 60 \quad \text{FV} = 1000$$

and CPT $\text{I} = .0892$, or 8.92%.

12. Bond 1: $P_1 = \dfrac{C_1 + Fr_1}{1.08} = \dfrac{C_1 + Fr_1}{1+s_1}$ and $s_1 = .08$, or 8%.

Bond 2: $P_2 = \dfrac{Fr_2}{1.08} + \dfrac{C_2 + Fr_2}{(1.08)^2} = \dfrac{Fr_2}{1+s_1} + \dfrac{C_2 + Fr_2}{\left(1+s_2\right)^2}$ and $s_2 = .08,$ or 8%.

Bond 3: $P_2 = \dfrac{Fr_3}{1.08} + \dfrac{Fr_3}{(1.08)^2} + \dfrac{C_3 + Fr_3}{(1.08)^3}$

$$= \frac{Fr_3}{1+s_1} + \frac{Fr_3}{\left(1+s_2\right)^2} + \frac{C_3 + Fr_3}{\left(1+s_3\right)^3} \text{ and } s_3 = .08, \text{ or } 8\%.$$

13. Consider a \$1 bond. We have

$$P = \frac{.08}{1.09} + \frac{.08}{(1.09)^2} + \frac{1.08}{(1.09)^3} = \frac{.08}{1.06} + \frac{.08}{(1.08)^2} + \frac{1.08}{(1+X)^3}$$

$$\text{or } .974687 = .144059 + \frac{1.08}{(1+X)^3}$$

and solving for $X = .0915,$ or 9.15%.

14. We are given $s_t = .09 - .02t,$ so that $s_1 = .07$ and $s_2 = .05.$

Bond A: $P_A = \dfrac{100}{1.07} + \dfrac{1100}{(1.05)^2} = \$1091.19$

and thus $1091.19 = \dfrac{100}{1+i_A} + \dfrac{1100}{\left(1+i_A\right)^2}.$

Solving the quadratic gives $i_A = .0509,$ or 5.09%.

Bond B: $P_B = \dfrac{50}{1.07} + \dfrac{1050}{(1.05)^2} = \$999.11$

and thus $999.11 = \dfrac{50}{1+i_B} + \dfrac{1050}{\left(1+i_B\right)^2}.$

Solving the quadratic gives $i_B = .0505,$ or 5.05%.

The yield rates go in the opposite direction than in Example 10.2.

15. (*a*) Applying formula (10.10)

$$\left(1+s_3\right)^3 = \left(1+s_1\right)\left(1+{}_2f_1\right)^2$$
$$(1.0875)^3 = (1.07)\left(1+{}_2f_1\right)^2$$

and solving ${}_2f_1 = .0964,$ or 9.64%.

(*b*) $(1+s_5)^5 = (1+s_2)^2(1+{}_3f_2)^3$

$(1.095)^5 = (1.08)^2(1+{}_3f_2)^3$

and solving ${}_3f_2 = .1051,$ or 10.51%.

16. (*a*) We have

$$\begin{aligned}(1+s_4)^4 &= (1+f_0)(1+f_1)(1+f_2)(1+f_3)\\ &= (1.09)(1.09)(1.86)(1.078) = 1.39092\end{aligned}$$

and solving $s_4 = .0860,$ or 8.60%.

(*b*) We have

$$\begin{aligned}(1+{}_3f_2)^3 &= (1+f_2)(1+f_3)(1+f_4)\\ &= (1.086)(1.078)(1.066) = 1.24797\end{aligned}$$

and solving ${}_3f_2 = .0766,$ or 7.66%.

17. We have

$$(1+s_2)^2 = (1+s_1)(1+f_1)$$
$$(1.06)^2 = (1.055)(1+f_1) \quad \text{and } 1+f_1 = 1.06502$$
$$(1+s_3)^3 = (1+s_1)(1+{}_2f_1)^2$$
$$(1.065)^3 = (1.055)(1+{}_2f_1)^2 \quad \text{and } (1+{}_2f_1)^2 = 1.14498$$
$$(1+s_4)^4 = (1+s_1)(1+{}_3f_1)^3$$
$$(1.07)^4 = (1.055)(1+{}_3f_1)^3 \quad \text{and } (1+{}_3f_1)^2 = 1.24246.$$

The present value of the 1-year deferred 3-year annuity-immediate is

$$\frac{1000}{1.06502} + \frac{1000}{1.14498} + \frac{1000}{1.24246} = \$2617.18.$$

18. We are given that

$$f_3 = .1076 \quad \text{and} \quad f_4 = .1051$$

and

$$\begin{aligned}(1+{}_2f_3)^2 &= (1+f_3)(1+f_4)\\ &= (1.1076)(1.1051)\end{aligned}$$

and solving ${}_2f_3 = .1063,$ or 10.63%.

19. We have

$$i = f_1 = \frac{(1+s_2)^2}{1+s_1} - 1 = \frac{(1.095)^2}{1.085} - 1 = .1051, \text{ or } 10.51\%.$$

20. We have

$$j = f_4 = \frac{(1+s_5)^5}{(1+s_4)^4} - 1 = \frac{(1.075)^5}{(1.082)^4} - 1 = .0474, \text{ or } 4.74\%.$$

21. The present value of this annuity today is

$$5000\left[\frac{1}{(1.0575)^2} + \frac{1}{(1.0625)^3} + \frac{1}{(1.0650)^4}\right] = 12,526.20.$$

The present value of this annuity one year from today is

$$12,526.20(1+s_1) = 12,526.20(1.05) = \$13,153 \text{ to the nearest dollar.}$$

22. We proceed as follows:

$$\begin{aligned}
(1+{}_4f_1)^4 &= \frac{(1+s_5)^5}{1+s_1} = \frac{(1.095)^5}{1.07} = 1.47125 \\
(1+{}_3f_2)^3 &= \frac{(1+s_5)^5}{(1+s_2)^2} = \frac{(1.095)^5}{(1.08)^2} = 1.34966 \\
(1+{}_2f_3)^2 &= \frac{(1+s_5)^5}{(1+s_3)^3} = \frac{(1.095)^5}{(1.0875)^3} = 1.22400 \\
(1+{}_1f_4)^2 &= \frac{(1+s_5)^5}{(1+s_4)^4} = \frac{(1.095)^5}{(1.0925)^4} = 1.10506.
\end{aligned}$$

We then evaluate $s_{\overline{5}|}$ as

$$s_{\overline{5}|} = 1.47125 + 1.34966 + 1.22400 + 1.10506 + 1 = 6.150.$$

23. For the one-year bond:

$$P = \frac{550}{1.07} = 514.02$$

so, no arbitrage possibility exists.

For the two-year bond:

$$P = \frac{50}{1.07} + \frac{550}{(1.08)^2} = 518.27$$

so, yes, an arbitrage possibly does exist.

Buy the two-year bond, since it is underpriced. Sell one-year \$50 zero coupon bond short for $50/1.07 = \$46.73$. Sell two-year \$550 zero coupon bond short for $550/(1.08)^2 = \$471.54$. The investor realizes an arbitrage profit of $46.73 + 471.54 - 516.00 = \$2.27$ at time $t = 0$.

24. (*a*) Sell one-year zero coupon bond at 6%. Use proceeds to buy a two-year zero coupon bond at 7%. When the one-year coupon bond matures, borrow proceeds at 7% for one year.

(*b*) The profit at time $t = 2$ is

$$1000(1.07)^2 - 1000(1.06)(1.07) = \$10.70.$$

25. The price of the 2-year coupon bond is

$$P = \frac{5.5}{1.093} + \frac{105.5}{(1.093)^2} = 93.3425.$$

Since the yield to maturity rate is greater than either of the two spot rates, the bond is underpriced.

Thus, buy the coupon bond for 93.3425. Borrow the present value of the first coupon at 7% for $5.5/1.07 = 5.1402$. Borrow the present value of the second coupon and maturity value at 9% for $105.5/(1.09)^2 = 88.7972$. There will be an arbitrage profit of $5.1402 + 88.7972 - 93.3425 = \$.59$ at time $t = 0$.

26. (*a*) Applying formula (10.17) we have

$$\begin{aligned}\lambda_t &= \frac{1}{t}\int_0^t \delta_r\,dr = \frac{1}{t}\int_0^t \left(.03 + .008r + .0018r^2\right)dr \\ &= \frac{1}{t}\left[.03r + .004r^2 + .0006r^3\right]_0^t \\ &= .03 + .004t + .0006t^2 \quad \text{for } 0 \le t \le 5.\end{aligned}$$

(*b*) Applying formula (10.13) we have

$$\begin{aligned}s_2 &= e^{\lambda_2} - 1 = e^{(.03+.008+.0024)} - 1 \\ &= .0412, \text{ or } 4.12\%.\end{aligned}$$

(*c*) Similar to (*b*)

$$s_5 = e^{\lambda_5} - 1 = e^{(.03+.02+.015)} - 1$$
$$= .06716.$$

Now

$$(1+s_5)^5 = (1+s_2)^2 (1+ {}_3 f_2)^3$$
$$(1.06716)^5 = (1.04123)^2 (1+ {}_3 f_2)^3$$

and solving ${}_3 f_2 = .0848,$ or 8.48%.

(*d*) We have

$$\frac{d\lambda_t}{dt} = .004 + .0012t > 0 \text{ for } t > 0$$

so we have a normal yield curve.

27. Applying formula (10.18) we have

$$\delta_t = \lambda_t + t\frac{d\lambda_t}{dt} = (.05 + .01t) + t(.01)$$
$$= .05 + .02t.$$

The present value is

$$a^{-1}(5) = e^{-\int_0^5 \delta_t dt} = e^{-\int_0^5 [.05+.02t]dt}$$
$$= e^{-[.05t+.01t^2]_0^5} = e^{-.25-.25} = e^{-.50}$$
$$= .6065.$$

Alternatively, λ_5 is a level continuous spot rate for $t = 5,$ i.e. $\lambda_5 = .05 + (.01)(5) = .1$.
We then have

$$a^{-1}(5) = e^{-5(.1)} = e^{-.5} = .6065.$$

28. Invest for three years with no reinvestment:

$$100,000(1.0875)^3 = 128,614.$$

Reinvest at end of year 1 only:

$$100,000(1.07)(1.10)^2 = 129,470.$$

Reinvest at end of year 2 only:

$$100,000(1.08)^2(1.11) = 129,470.$$

Reinvest at end of both years 1 and 2:

$$100,000(1.07)(1.09)(1.11) = 129,459.$$

29. Year 1: $1+i=(1+i')(1+r_1)$

$1.07=(1.03)(1+r_1)$ and $r_1=.03883$, or 3.9%.

Year 2: $(1+i)^2=(1+i')^2(1+r_1)(1+r_2)$

$(1.08)^2=(1.03)^2(1.03883)(1+r_2)$ and $r_2=.05835$, or 5.8%.

Year 3: $(1+i)^3=(1+i')^3(1+r_1)(1+r_2)(1+r_3)$

$(1.0875)^3=(1.03)^3(1.03883)(1.05835)(1+r_3)$ and $r_3=.07054$, or 7.1%.