Chapter 11

1. A generalized version of formula (11.2) would be

$$\bar{d} = \frac{t_1 v^{t_1} R_{t_1} + t_2 v^{t_2} R_{t_2} + \cdots + t_n v^{t_n} R_{t_n}}{v^{t_1} R_{t_1} + v^{t_2} R_{t_2} + \cdots + v^{t_n} R_{t_n}}$$

where $0 < t_1 < t_2 < \ldots < t_n$. Now multiply numerator and denominator by $(1+i)^{t_1}$ to obtain

$$\bar{d} = \frac{t_1 R_{t_1} + t_2 v^{t_2 - t_1} R_{t_2} + \cdots + t_n v^{t_n - t_1} R_{t_n}}{R_{t_1} + v^{t_2 - t_1} R_{t_2} + \cdots + v^{t_n - t_1} R_{t_n}}.$$

We now have $\lim_{i \to \infty} \bar{d} = \lim_{v \to 0} \bar{d} = \frac{t_1 R_{t_1}}{R_{t_1}} = t_1.$

2. We can apply the dividend discount model and formula (6.28) to obtain

$$P(i) = D(i-k)^{-1}.$$

We next apply formula (11.4) to obtain

$$\bar{v} = -\frac{P'(i)}{P(i)} = \frac{D(i-k)^{-2}}{D(i-k)^{-1}}$$
$$= (i-k)^{-1} = (.08 - .04)^{-1}.$$

Finally, we apply formula (11.5)

$$\bar{d} = \bar{v}(1+i) = \frac{1.08}{.08 - .04} = 27.$$

3. We can use a continuous version for formula (11.2) to obtain

$$\bar{d} = \frac{\int_0^n t v^t dt}{\int_0^n t v^t dt} = \frac{(\bar{I}\,\bar{a})_{\overline{n|}}}{a_{\overline{n|}}}$$

and then apply formula (11.5)

$$\bar{v} = \frac{\bar{d}}{1+i} = \frac{v(\bar{I}\,\bar{a})_{\overline{n|}}}{\bar{a}_{\overline{n|}}}.$$

4. The present value of the perpetuity is

$$a_{\overline{\infty}|} = \frac{1}{i}.$$

The modified duration of the perpetuity is

$$\bar{v} = \frac{\bar{d}}{1+i} = \frac{v\sum_{t=1}^{\infty} tv^t}{\sum_{t=1}^{\infty} v^t} = \frac{v(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}}$$

$$= \frac{v/id}{1/i} = \frac{v}{d} = \frac{v}{iv} = \frac{1}{i}.$$

5. Applying the fundamental definition we have

$$\bar{d} = \frac{10(Ia)_{\overline{8}|} + 800v^8}{10a_{\overline{8}|} + 100v^8} \quad \text{at } i = 8\%$$

$$= \frac{(10)(23.55274) + 800(.54027)}{(10)(5.74664) + 100(.54027)} = 5.99.$$

6. (*a*) We have $\bar{v} = -\dfrac{P'(i)}{P(i)} = \dfrac{\bar{d}}{1+i}$

so that $\dfrac{650}{100} = \dfrac{\bar{d}}{1.07}$ and $\bar{d} = 6.955.$

(*b*) We have $P(i+h) \approx P(i)[1 - h\bar{v}]$

so that $P(.08) \approx P(.07)[1 - .01\bar{v}]$

$$= 100[1 - (.01)(6.5)] = 93.50.$$

7. Per dollar of annual installment payment the prospective mortgage balance at time $t = 3$ will be $a_{\overline{12}|.06} = 8.38384$. Thus, we have

$$\bar{d} = \frac{\sum tv^t R_t}{\sum v^t R_t} = \frac{(1.06)^{-1} + 2(1.06)^{-2} + 3(9.38384)(1.06)^{-3}}{(1.06)^{-1} + 2(1.06)^{-2} + 9.38384(1.06)^{-3}}$$

$$= \frac{26.359948}{9.712246} = 2.71.$$

8. We have $P(i) = R(1 + i)^{-1}$

$$P'(i) = -R(1 + i)^{-2}$$
$$P''(i) = 2R(1 + i)^{-3}$$

and $\bar{c} = \dfrac{P''(i)}{P(i)} = \dfrac{2R(1+i)^{-3}}{R(1+i)^{-1}} = 2(1+i)^{-2} = \dfrac{2}{(1.08)^2} = 1.715.$

9. (*a*) Rather than using the definition directly, we will find the modified duration first and adjust it, since this information will be needed for part (*b*). We have

$$P(i) = 1000\left[(1+i)^{-1} + (1+i)^{-2}\right]$$
$$P'(i) = 1000\left[-(1+i)^{-2} - 2(1+i)^{-3}\right]$$
$$P''(i) = 1000\left[2(1+i)^{-3} + 6(1+i)^{-4}\right].$$

Now, $\bar{v} = -\dfrac{P'(i)}{P(i)} = \dfrac{(1.1)^{-2} + 2(1.1)^{-3}}{(1.1)^{-1} + (1.1)^{-2}} = \dfrac{1.1+2}{(1.1)^2 + 1.1}$

and $\bar{d} = \bar{v}(1+i) = \dfrac{1.1+2}{1.1+1} = \dfrac{3.1}{2.1} = 1.48.$

(*b*) We have

$$\bar{c} = \frac{P''(i)}{P(i)} = \frac{2(1+i)^{-3} + 6(1+i)^{-4}}{(1+i)^{-1} + (1+i)^{-2}}$$

and multiplying numerator and denominator by $(1+i)^4$

$$\frac{2(1+i)+6}{(1+i)^3 + (1+i)^2} = \frac{2(1.1)+6}{(1.1)^3 + (1.1)^2} = \frac{8.2}{2.541} = 3.23.$$

10. When there is only one payment $\bar{d}$ is the time at which that payment is made for any force of interest. Therefore, $\dfrac{d\bar{d}}{d\delta} = \dfrac{d\bar{v}}{d\delta} = \sigma^2 = 0.$

11. (*a*) $P(i) = 1000\left[(1+i)^{-1} + 2(1+i)^{-2} + 3(1+i)^{-3}\right]$

$$= 1000\left[(1.25)^{-1} + 2(1.25)^{-2} + 3(1.25)^{-3}\right] = \$3616.$$

(*b*) $\bar{d} = \dfrac{1000\left[(1.25)^{-1} + 4(1.25)^{-2} + 9(1.25)^{-3}\right]}{1000\left[(1.25)^{-1} + 2(1.25)^{-2} + 3(1.25)^{-3}\right]} = \dfrac{7968}{3616} = 2.2035.$

(*c*) $\bar{v} = \dfrac{\bar{d}}{1+i} = \dfrac{2.2035}{1.25} = 1.7628.$

(*d*) $\bar{c} = \dfrac{P''(i)}{P(i)} = \dfrac{2(1+i)^{-3} + 12(1+i)^{-4} + 36(1+i)^{-5}}{(1+i)^{-1} + 2(1+i)^{-2} + 3(1+i)^{-3}}$

$$= \frac{2(1.25)^{-3} + 12(1.25)^{-4} + 36(1.25)^{-5}}{3.616} = 4.9048.$$

12. Per dollar of installment payment, we have

$$P(i) = (1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-n}$$
$$P''(i) = (1)(2)(1+i)^{-3} + (2)(3)(1+i)^{-4} + \cdots + (n)(n+1)(1+i)^{-n-2}.$$

If $i = 0,$ the convexity is

$$\bar{c} = \frac{P''(0)}{P(0)} = \frac{1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)}{1+1+\cdots+1}$$
$$= \frac{(1^2 + 2^2 + \cdots + n^2) + (1+2+\cdots+n)}{n}$$
$$= \frac{\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)}{n} = \frac{n(n+1)(2n+1) + 3n(n+1)}{6n}$$
$$= \frac{n(n+1)(2n+4)}{6n} = \frac{1}{3}(n+1)(n+2).$$

13. We have $P(i) = D(i-k)^{-1}$

$$P''(i) = 2D(i-k)^{-3}$$

so that $\bar{c} = \dfrac{P''(i)}{P(i)} = \dfrac{2D(i-k)^{-3}}{D(i-k)^{-1}} = \dfrac{2}{(i-k)^2} = \dfrac{2}{(.08-.04)^2} = 1250.$

14. From formula (11.10)

$$\frac{d\bar{v}}{di} = \bar{v}^2 - \bar{c} \quad \text{or} \quad -800 = (6.5)^2 - \bar{c} \quad \text{or} \quad \bar{c} = 800 + (6.5)^2 = 842.25.$$

Now applying formula (11.9b), we have

$$P(i+h) \approx P(i)\left[1 - h\bar{v} + \frac{h^2}{2}\bar{c}\right]$$

$$P(.08) \approx 100\left[1-(.01)(6.5)+\frac{(.01)^2}{2}(842.25)\right]$$

$$= 97.71.$$

15. (*a*) From formula (11.19)

$$\bar{d}_e = \frac{P(i-h)-P(i+h)}{2hP(i)} = \frac{101.6931-100.8422}{2(.001)(101.2606)}$$

$$= 4.20.$$

(*b*) From formula (11.20)

$$\bar{c}_e = \frac{P(i-h)-2P(i)+P(i+h)}{h^2P(i)}$$

$$= \frac{101.6931-2(101.2606)+100.8422}{(.001)^2(101.2606)} = 139.24.$$

(*c*) From formula (11.22)

$$P(i+h) \approx P(i)\left[1 - h\bar{d}_e + \frac{h^2}{2}\bar{c}_e\right]$$

$$= 101.2606\left[1-(.0075)(4.20)+\frac{(.0075)^2}{2}(139.24)\right]$$

$$= \$98.47.$$

16. We have:

$$P(.09) = \frac{100,000}{a_{\overline{20|}.08}}\left[a_{\overline{10|}.08} + (1.08)^{-10} a_{\overline{10|}.09}\right]$$

$$= 98,620.43.$$

$$P(.08) = 100,000.00.$$

$$P(.07) = \frac{100,000}{a_{\overline{20|}.08}} a_{\overline{10|}.08} + \frac{\left[100,000(1.08)^{10} - \frac{100,000}{a_{\overline{20|}.08}} s_{\overline{10|}.08}\right] a_{\overline{5|}.07}(1.08)^{-10}}{a_{\overline{5|}.08}}$$

$$= 100,852.22.$$

(*a*) We have

$$\bar{d}_e = \frac{P(i-h)-P(i+h)}{2hP(i)} = \frac{100,852.22-98,620.43}{2(.01)(100,000)} = 1.12$$

(*b*) We have

$$\bar{c}_e = \frac{P(i-h)-2P(i)+P(i+h)}{h^2P(i)}$$
$$= \frac{100,852.22-2(100,000)+98,620.43}{(.01)^2(100,000)} = -52.73$$

17. Using formula (11.22)

$$P(.09) \approx 100,000\left[1-(.01)(1.116)+\tfrac{(.01)^2}{2}(-52.734)\right] = \$98,620$$

which agrees with the price calculated in Exercise 16.

$$P(.07) \approx 100,000\left[1-(.01)(1.116)+\tfrac{(.01)^2}{2}(-52.734)\right] = \$100,852$$

which agrees with the price calculated in Exercise 16.

18. We know that $\bar{c} = \dfrac{P''(i)}{P(i)}$. Let $\Delta i = h.$

Then $P'(i) \approx \dfrac{\Delta P(i)}{\Delta i}$, so that

$$P''(i) \approx \frac{\Delta}{\Delta i}\frac{\Delta P(i)}{\Delta i} = \frac{\Delta^2 P(i)}{(\Delta i)^2}$$
$$= \frac{[P(i+h)-P(i)]-[P(i)-P(i-h)]}{(\Delta i)^2}$$

and $\bar{c} = \dfrac{P(i-h)-2P(i)+P(i+h)}{h^2P(i)}$.

19. Directly form formula (11.24), we have

$$\frac{(21.46)(980)+(12.35)(1015)+(16.67)(1000)}{980+1015+1000}$$
$$=16.77.$$

20. (*a*) Time 1 before payment is made:

$$\bar{d} = \frac{(0)(1)+(1)(1.1)^{-1}+(2)(1.1)^{-2}}{1+(1.1)^{-1}+(1.1)^{-2}} = .9366.$$

Time 1 after payment is made:

$$\bar{d} = \frac{(1)(1.1)^{-1}+(2)(1.1)^{-2}}{(1.1)^{-1}+(1.1)^{-2}} = 1.4762.$$

"Jump" $= 1.4762 - .9366 = .540.$

(*b*) Time 2 before payment is made:

$$\bar{d} = \frac{(0)(1)+(1)(1.1)^{-1}}{1+(1.1)^{-1}} = .4762.$$

Time 2 after payment is made:

$$\bar{d} = \frac{(1)(1.1)^{-1}}{(1.1)^{-1}} = 1.0000$$

"Jump" $= 1.0000 - .4762 = .524.$

(*c*) The numerator is the same before and after the "jump." The denominator is one less after the jump than before. The effect is greater when the numerator is greater.

21. Treasury bills have a stated rate at simple discount, which can be considered to be a discount rate convertible quarterly as they rollover from quarter to quarter. We have

$$\left(1-\frac{d^{(4)}}{4}\right)^{-2} = 1+\frac{i^{(2)}}{2}$$

$$\left(1-\frac{.06}{4}\right)^{-2} = 1+\frac{i^{(2)}}{2} \qquad i^{(2)} = .0613775.$$

Run tests at $i^{(2)} = .0513775$ and $.0713775$.

$$\left(1-\frac{d^{(4)}}{4}\right)^{-2} = 1+\frac{.0513775}{2}, \text{ so } d_L^{(4)} = .0504.$$

$$\left(1-\frac{d^{(4)}}{4}\right)^{-2} = 1+\frac{.0713775}{2}, \text{ so } d_H^{(4)} = .0695.$$

Thus, use 5.04% and 6.95% rates of discount.

22. Macaulay convexity equals to the square of Macaulay duration for single payments. Thus, we have

	Macaulay Duration	Macaulay Convexity	Amount
Bond 1	2	4	10,000
Bond 2	5	25	20,000
Bond 3	10	100	30,000

Then applying formula (11.25)

$$\frac{10,000(4)+20,000(25)+30,000(100)}{10,000+20,000+30,000}=59.$$

23. We set

$$CF_0=-2,948,253$$
$$CF_1=1,105,383$$
$$CF_2=1,149,598$$
$$CF_3=1,195,582$$

and obtain $\text{IRR}=8.18\%$ using a financial calculator.

24. We have

$$\frac{1,105,383}{1.0875}+\frac{1,149,598}{(1.08)^2}+\frac{1,195,582}{(1.07)^3}=\$2,977,990.$$

25. Using absolute matching with zero-coupon bonds, we have

$$\frac{1000}{1.1}+\frac{2000}{(1.12)^2}=\$2503.48.$$

26. Let F_1 and F_2 be the face amount of 1-year and 2-year bonds. At the end of the second year

$$F_2+.06F_2=10,000 \quad \text{and} \quad F_2=9433.96.$$

At the end of the first year

$$(.06)(9433.96)+F_1+.04F_1=10,000 \quad \text{and} \quad F_1=9071.12.$$

The price of the 1-year bond is

$$\frac{9071.12(104)}{105} = 8984.73.$$

The price of the 2-year bond is

$$9433.96\left[\frac{.06}{1.05} + \frac{1.06}{(1.05)^2}\right] = 9609.38.$$

The total price is $8984.73 + 9609.38 = \$18,594$ to the nearest dollar.

27. (*a*) The amount of Bond B is

$$\frac{2000}{1.025} = 1951.220$$

in order to meet the payment due in one year. The amount of Bond A is

$$\frac{1000 - 1951.220(.025)}{1.04} = 914.634$$

in order to meet the payment due in 6 months. The cost of Bond B is

$$P_B = 1951.220\left[\frac{.025}{1.035} + \frac{1.025}{(1.035)^2}\right] = 1914.153.$$

The cost of Bond A is

$$P_A = 914.634\left(\frac{1.04}{1.03}\right) = 923.514.$$

Thus, the total cost to the company is

$$1914.153 + 923.514 = \$2837.67.$$

(*b*) The equation of value is

$$2837.67 = 1000(1+j)^{-1} + 2000(1+j)^{-2}$$

which is a quadratic. Solving the quadratic gives $j = .03402,$ which is a semiannual interest rate. Thus, the nominal IRR convertible semiannually is $2j = 2(.03402) = .0680,$ or 6.80%.

28. (*a*) The liability is a single payment at time $t = 1,$ so $\bar{d} = 1$. We then have

$$\bar{v} = \frac{\bar{d}}{1+i} = \frac{1}{1.10} = .090909$$

which is equal to the modified duration of the assets.

(*b*) We have

$$P(i) = 1100(1+i)^{-1}$$
$$P''(i) = 2200(1+i)^{-3}.$$

Thus, the convexity of the liability is

$$\bar{c} = \frac{P''(.10)}{P(.10)} = \frac{2200(1.1)^{-3}}{1100(1.1)^{-1}}$$
$$= \frac{2}{(1.1)^2} = 1.65289$$

which is less than the convexity of the assets of 2.47934.

29. (*a*) $P(i) = 600 + (400)(1.21)(1+i)^{-2} - 1100(1+i)^{-1}$

(1) $P(.09) = 600 + 484(1.09)^{-2} - 1100(1.09)^{-1} = -1.8012$

(2) $P(.10) = 600 + 484(1.10)^{-2} - 1100(1.10)^{-1} = 0$

(3) $P(.11) = 600 + 484(1.11)^{-2} - 1100(1.11)^{-1} = 1.8346$

(*b*) $P(i) = 400 + (600)(1.21)(1+i)^{-2} - 1100(1+i)^{-1}$

(1) $P(.09) = 400 + 726(1.09)^{-2} - 1100(1.09)^{-1} = 1.8854$

(2) $P(.10) = 400 + 726(1.10)^{-2} - 1100(1.10)^{-1} = 0$

(3) $P(.11) = 400 + 726(1.11)^{-2} - 1100(1.11)^{-1} = -1.7531$

(*c*) In Example 11.14 $P(i) > 0$ for a 1% change in *i* going in either direction, since the portfolio is immunized with 500 in each type of investment. If the investment allocation is changed, the portfolio is no longer immunized.

30. (*a*) From formula (11.27) we have

$$P(i) = A_1(1+i)^{-1} + A_5(1+i)^{-5} - 100\left[(1+i)^{-2} + (1+i)^{-4} + (1+i)^{-6}\right].$$

Now multiplying by $(1+i)^5$ and setting $i = .1$

$$A_1(1.1)^4 + A_5 = 100\left[(1.1)^3 + 1.1 + (1.1)^{-1}\right]$$

or $1.4641A_1 + A_5 = 334.01.$

From formula (11.28) we have

$$P'(i) = -A_1(1+i)^{-2} - 5A_5(1+i)^{-6} + 100\left[2(1+i)^{-3} + 4(1+i)^{-5} + 6(1+i)^{-7}\right].$$

Now multiplying by $(1+i)^6$ and setting $i=.1$

$$A_1(1.1)^4+5A_5=100\left[2(1.1)^3+4(1.1)+6(1.1)^{-1}\right]$$

or $1.4641A_1+5A_5=1251.65.$

Solving two equations in two unknowns, we have $A_1=\$71.44$ and $A_5=\$229.41$

(*b*) Testing formula (11.29) we have

$$P''(i)=2A_1(1+i)^{-3}+30A_5(1+i)^{-7}-100\left[6(1+i)^{-4}+20(1+i)^{-6}+42(1+i)^{-8}\right]$$

and

$$\begin{aligned}P''(.10)&=(2)(71.44)(1.1)^{-3}+(30)(229.41)(1.1)^{-7}\\&\quad-100\left[(6)(1.1)^{-4}+(20)(1.1)^{-6}+(42)(1.1)^{-8}\right]\\&=140.97>0.\end{aligned}$$

Yes, the conditions for Redington immunization are satisfied.

31. Adapting formulas (11.30) and (11.31) to rates of interest, we have:

$$A(1.1)^5+B(1.1)^{-5}-10,000=0$$
$$5A(1.1)^5-5B(1.1)^{-5}=0$$

Solving two equations in two unknowns give the following answers:

(*a*) $A=\$3104.61.$

(*b*) $B=\$8052.56.$

32. Again adapting formulas (11.30) and (11.31) to rates of interest, we have:

$$A(1.1)^a+6000(1.1)^{-2}-10,000=0$$
$$aA(1.1)^a-6000(2)(.1)^{-2}=0$$

or

$$aA(1.1)^a=9917.36$$
$$A(1.1)^a=5041.32.$$

Solving two equations in two unknowns gives the following answers:

(*a*) $A=5041.32(1.1)^{-1.96721}=\$4179.42.$

(*b*) $a=\dfrac{9917.36}{5041.32}=1.967$.

33. (*a*) If $f = .075$ and $k_1 = .10$, we have

$$A_2 = -.016225 p_1 + .011325 > 0$$

or $p_1 < .6980$.

If $f = .09$ and $k_1 = .90$, we have

$$A_2 = -.000025 p_1 + .015325 > 0$$

or $p_1 < .6130$.

Thus, choose p_1 so that $0 < p_1 < .6980$.

(*b*) If $f = .065$ and $k_1 = .10$, we have

$$A_2 = -.027025 p_1 + .012325 > 0$$

or $p_1 < .4561$.

If $f = .10$ and $k_1 = .90$, we have

$$A_2 = -.010775 p_1 - .007175 > 0$$

or $p_1 > .6613$.

Thus, no solution exists.

34. (*a*) If $f = .07$ and $k_1 = .20$, we have

$$A_2 = -.021625 p_1 + .012825 > 0$$

or $p_1 < .5931$.

If $f = .095$ and $k_1 = .80$, we have

$$A_2 = .005375 p_1 - .001175 > 0$$

or $p_1 > .2186$.

Thus, choose p_1 so that $.2186 < p_1 < .5931$.

(*b*) If $f = .07$ and $k_1 = 0$, we have

$$A_2 = -.021625 p_1 + .010825 > 0$$

or $p_1 < .5006$.

If $f = .095$ and $k_1 = 1$, then

$$A_2 = .005375 p_1 - .004175 > 0$$

or $p_1 > .7767$.

Thus, no solution exists.

35. The present value of the liability at 5% is

$$1,000,000(1.05)^{-4} = 822,703.$$

The present value of the bond at 5% is 1,000,000.

If interest rates decrease by ½%, the coupons will be reinvested at 4.5%. The annual coupon is $822,703(.05) = 41,135$. The accumulated value 12/31/z+4 will be

$$822,703 + 41,135 s_{\overline{4}|.045} = 998,687.$$

The liability value at that point is 1,000,000 creating a loss of 1,000,000-998,687 = $1313.

If the interest rates increase by ½%, the accumulated value 12/31/z+4 will be

$$822,703 + 41,135 s_{\overline{4}|.055} = 1,001,323$$

creating a gain of $1,001,323 - 1,000,000 = \$1323$.

36. (*a*) Under Option A, the 20,000 deposit grows to $20,000(1.1) = 22,000$ at time $t = 1$. Half is withdrawn, so that 11,000 continues on deposit and grows to $11,000(1.1) = 12,100$ at time $t = 2$. The investment in two-year zero coupon bonds to cover this obligation is $12,100(1.11)^{-2} = 9802.63$. Thus, the profit at inception is $10,000 - 9802.63 = \$179.37$.

(*b*) Using the principles discussed in Chapter 10, we have

$$(1.11)^2 = (1.10)(1+f), \text{ so that}$$

$$f = \frac{(1.11)^2}{1.10} = .1201, \text{ or } 12.01\%.$$

37. The present value the asset cash inflow is

$$P_A(i) = 35,000(1.08)^5(1+i)^{-5} + (.08)(50,000)(1/i).$$

The present value the liability cash outflow is

$$P_L(i) = 85,000(1.08)^{10}(1+i)^{-10}.$$

We then have the following derivatives:

$$P_A'(i) = -5(35,000)(1.08)^5(1+i)^{-6} - 4000i^{-2}$$
$$P_L'(i) = -10(85,000)(1.08)^{10}(1+i)^{-11}$$
$$P_A''(i) = (5)(6)(35,000)(1.08)^5(1+i)^{-7} + 8000i^{-3}$$
$$P_L''(i) = (10)(11)(85,000)(1.08)^{10}(1+i)^{-12}.$$

If $i = .08$, we have the following:

$$P_A(.08) = 85,000 \text{ and } P_B(.08) = 85,000$$
$$\bar{v}_A = -\frac{P_A'(.08)}{P_A(.08)} = \frac{787,037.04}{85,000.00} = 9.2593$$
$$\bar{v}_L = -\frac{P_L'(.08)}{P_L(.08)} = \frac{787,037.04}{85,000.00} = 9.2593$$
$$\bar{c}_A = -\frac{P_A''(.08)}{P_A(.08)} = \frac{16,675,026}{85,000} = 196.18$$
$$\bar{c}_L = -\frac{P_L''(.08)}{P_L(.08)} = \frac{8,016,118}{85,000} = 94.31.$$

Thus, the investment strategy is optimal under immunization theory, since

(1) $P_A(.08) = P_B(.08)$
(2) $\bar{v}_A = \bar{v}_L$
(3) $\bar{c}_A > \bar{c}_L$

38. This is a lengthy exercise and a complete solution will not be shown. The approach is similar to Exercise 37. A sketch of the full solution appears below.

When the initial strategy is tested, we obtain the following:

$$P_A(.10) = 37,908 \qquad \bar{v}_A = 2.7273$$
$$P_L(.10) = 37,908 \qquad \bar{v}_L = 2.5547$$

Since $\bar{v}_A \neq \bar{v}_L$, the strategy cannot be optimal under immunization theory.

The superior strategy lets *x*, *y*, *z* be amounts invested in 1-year, 3-year, 5-year bonds, respectively. The three immunization conditions are set up leading to two equations and one inequality in three unknowns.

The solution $x = \$13,223$
$y = \$15,061$
$z = \$9624$ satisfies these three conditions.

39. (*a*) We have

$$\bar{d} = \frac{i(v + 2v^2 + 3v^3 + \cdots + nv^n) + nv^n}{i(v + v^2 + v^3 + \cdots + v^n) + v^n}$$

$$= \frac{i(Ia)_{\overline{n}|} + nv^n}{ia_{\overline{n}|} + v^n} = \frac{\ddot{a}_{\overline{n}|} - nv^n + nv^n}{1 - v^n + v^n} = \ddot{a}_{\overline{n}|}.$$

(*b*) We have $\ddot{a}_{\overline{10}|.08} = 7.25$, as required.

40. (*a*) We have

$$P(i+h) \approx P(i)\left(\frac{1+i}{1+i+h}\right)^{\bar{d}}$$

so that

$$P(.09) \approx P(.08)\left(\frac{1.08}{1.09}\right)^{\bar{d}} = (1)\left(\frac{1.08}{1.09}\right)^{7.2469} = .9354.$$

(*b*) The error in this approach is

$$.9358 - .9354 = .0004\,.$$