22

Graph Multicolorings

Author: Kenneth H. Rosen, AT&T Laboratories.

Prerequisites: The prerequisites for this chapter are basic graph terminology and graph colorings. See Sections 9.1, 9.2, and 9.8 of *Discrete Mathematics and Its Applications*.

Introduction

Graph coloring has been studied for many years and has been used in many different applications. In particular, graph colorings have been employed for over a hundred years to study map colorings. With each map we associate a graph — this graph has a vertex for each region of the map and an edge connecting two vertices if and only if the regions they represent share a common boundary. A coloring of the map so that no two regions with a common boundary have the same color corresponds to a coloring of the vertices of the associated graph so that no two adjacent vertices are assigned the same color. This traditional application of graph colorings has been one of the major factors in the development of graph theory.

However, there are many modern applications of graph colorings not involving maps. Some of these applications are task scheduling, meeting scheduling, maintenance scheduling, channel assignment for television stations, and frequency assignment for mobile telephones. To model variations of these applications we need various generalizations of graph colorings. In this chapter we will study these applications of graph colorings. We will make several different generalizations of graph colorings that are required to model variations of these problems.

Many of the generalizations of graph colorings that we will study in this chapter have been introduced and studied by Fred Roberts. More information about the topics covered, as well as the description of other applications, including traffic phasing and garbage collection, can be found in Roberts' survey article [5].

Applications of Graph Colorings

In this section we will describe a variety of applications of graph colorings. (Additional descriptions of these and other applications of graph colorings can be found in Section 9.8 of *Discrete Mathematics and Its Applications*, and in [4].)

Examination Scheduling Consider the problem of scheduling 25 final examinations in the Computer Science Department of a college. The examinations need to be scheduled so that no student has two final exams at the same time and no professor has final exams for two different courses at the same time. To model this problem, we build a graph where a vertex is associated to each course and an edge connects two vertices if the courses they represent cannot have final examinations at the same time. In a coloring of the associated graph, colors represent time slots. In order to minimize the number of time slots needed to schedule all exams, we find the chromatic number of this graph (i.e., the minimum number of colors needed to color the vertices so no adjacent vertices have the same color).

Task Scheduling Consider the problem of scheduling different tasks, some of which cannot be done at the same time. For example, a automobile repair shop may have 15 cars to repair, but only one machine used in a particular type of repair. To model this problem, we build a graph where a vertex is associated with each task and an edge connects two vertices if the tasks they represent are incompatible, that is, they cannot be done at the same time. For example, vertices representing car repairs that require the use of the same machine are joined by an edge. We use colors to represent time periods. Assuming that all tasks take the same time, to minimize the total time to complete the tasks we find the smallest number of colors needed to color the vertices of this graph. We find the chromatic number of this graph to minimize the time required to complete the tasks.

Meeting Scheduling Consider the problem of scheduling the meetings of different committees where some individuals serve on more than one committee. Clearly, no two committees can meet at the same time if they have a common member. Analogous to exam scheduling, to model this problem we build a graph where a vertex is associated to each committee and an edge connects two vertices if the committees they represent have a common member. In a coloring of this graph, colors represent time slots. We find the chromatic number of this graph to schedule all meetings in the least time.

Maintenance Scheduling A facility with repair bays is used to maintain a fleet of vehicles. Each vehicle is scheduled for regular maintenance during a time period and is assigned a repair bay. Two vehicles cannot be assigned the same space if they are scheduled for maintenance at overlapping times. To model this problem, we build a graph with vertices representing the vehicles where an edge connects two vertices if the vehicles they represent are assigned the same space for repair. We find the chromatic number of the associated graph to schedule maintenance for all vehicles using the smallest number of periods.

Channel or Frequency Assignment Two stations (radio, television, mobile telephone, etc.) that interfere, because of proximity, geography, power or some other factor, cannot be assigned the same channel. To model this problem we build a graph with vertices that represent the stations with an edge connecting two vertices if the stations they represent interfere. We find the chromatic number of the associated graph to assign frequencies to all stations using the smallest number of channels. (This model was first described by E. Gilbert in 1972 in unpublished work at Bell Laboratories.)

Many complications arise in these and other applications that make it necessary to apply generalizations of graph colorings. Using variations of some of these applications as motivation, we will now introduce several ways to generalize the notion of a graph coloring.

Graph Multicolorings

Suppose that there are two parts of each final examination at a college. Each part of an exam takes a morning or an afternoon. Therefore, we need to assign two time periods to each final. To model this problem, we need to assign *two* colors to each vertex of the associated graph, where each color represents a time period (such as Tuesday afternoon), so that no adjacent vertices are assigned a common color. This means that no student or faculty member is required to be at two final exams at the same time.

Consider the following variation of the channel assignment problem. Suppose we need to assign three channels to each station. For instance, backup channels may be needed when a channel is unusable; a station may need three channels to simultaneously broadcast three different programs; or a mobile radio station may need three different channels for teleconferencing. To model this problem, we need to assign *three* colors to each vertex of the associated graph, where each color represents a channel, so that no adjacent vertices are assigned a common color. This means that no stations that interfere will broadcast over the same channel.

These applications lead to the following definition.

Definition 1 An assignment of n colors to each vertex in a graph $G = (V, E)$ is called an *n-tuple coloring* or *multicoloring* if no two adjacent vertices are assigned a common color. □

We often want to minimize the total number of colors used in an n-tuple coloring. For instance, we want to schedule two-part final examinations using the fewest periods. Similarly, we want to assign three channels to each station using the fewest channels.

Definition 2 The *n-tuple chromatic number* of a graph G, denoted by $\chi_n(G)$, is the minimum number of colors required for an n-tuple coloring of G. □

For every graph G we have $\chi_n(G) \leq n\chi(G)$, where $\chi(G)$ is the chromatic number of G. To see this, note that we can construct an n-tuple coloring of G from a coloring of G by associating to each color n distinct new colors, with no new color associated to more than one of the original colors. As the next two examples illustrate, the equality $\chi_n(G) = n\chi(G)$ holds for some graphs, while for other graphs $\chi_n(G) < n\chi(G)$. Later we will see that equality holds for a large class of graphs.

Example 1 Find $\chi_2(K_4)$ and $\chi_3(K_4)$, where K_4 is the complete graph on four vertices.

Solution: Since every vertex of K_4 is adjacent to every other vertex, no color can be used more than once. Hence, a total of $4n$ colors are required for an n-tuple coloring of K_4. It therefore follows that $\chi_2(K_4) = 2 \cdot 4 = 8$ and $\chi_3(K_4) = 3 \cdot 4 = 12$. □

Example 2 Find $\chi_2(C_4)$ and $\chi_2(C_5)$, where C_n is the n-cycle graph.

Solution: As Figure 1 shows, we can construct a 2-tuple coloring of C_4 using four colors. Since we must use at least four colors so that two colors are assigned to each of two adjacent vertices, no fewer than four colors can be used. Hence $\chi_2(C_4) = 4$.

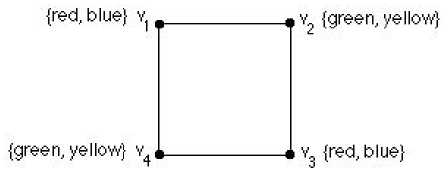


Figure 1. A 2-tuple coloring of C_4 using four colors.

In Figure 2 we display a 2-tuple coloring of C_5 using just five colors (rather than six colors that would be used if we started with a coloring of C_5 and then assigned two new colors to each of the original colors in this coloring). Since it is impossible to construct a 2-tuple coloring of C_5 using four colors (as the reader can easily demonstrate), it follows that $\chi_2(C_5) = 5$. □

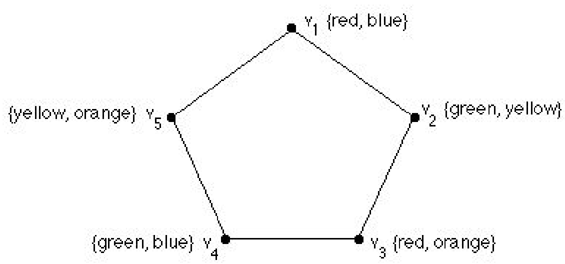


Figure 2. A 2-tuple coloring of C_5 using five colors.

Although $\chi_n(G)$ may be less than $n\chi(G)$, there is an important class of graphs for which these quantities are equal. Before describing these graphs, we first give some definitions.

Definition 3 A *clique* in a graph G is a subgraph of G that is a complete graph. □

Example 3 Find all cliques with four or more vertices in the graph G in Figure 3.

Solution: The subgraph containing the vertices a, b, d, and f, and the edges in G connecting these vertices, is a clique with four vertices. So is the subgraph containing the vertices b, d, e, and f, and the edges in G connecting these vertices. Inspection of the graph shows that there are no larger cliques and no other cliques with four vertices. □

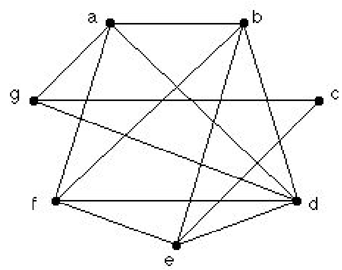


Figure 3. The graph G.

Before describing a class of graphs for which the n-tuple chromatic number is n times the chromatic number, we need the following definition.

Definition 4 The *clique number* of a graph G, denoted by $\omega(G)$, is the largest integer k such that G contains a clique of k vertices. □

Example 4 Find the clique number of the graph G in Figure 3.

Solution: By Example 3 we see that $\omega(G)$, the clique number of G, is 4 since G contains a clique of this size but no clique of a larger size. □

The chromatic number of a graph must be at least as large as the clique number; that is, $\omega(G) \leq \chi(G)$. This is true since any coloring of G will require $\omega(G)$ colors for the vertices in the largest clique. The other vertices in G may require additional colors. The special class of graphs we will be interested in have their clique numbers equal to their chromatic numbers, as described in the following definition.

Definition 5 A graph G is called *weakly γ-perfect* if its chromatic number equals its clique number, that is, if $\omega(G) = \chi(G)$. □

Example 5 Is the graph G is Figure 3 weakly γ-perfect?

Solution: Coloring the vertices a and e red, b and c blue, d green, and f and g yellow produces a 4-coloring of G. Since any coloring of G uses at least four colors (since a, b, d, and f form a complete subgraph), we conclude that $\chi(G) = 4$. By Example 4 we know that $\omega(G) = 4$. Since $\chi(G) = \omega(G)$, it follows that G is weakly γ-perfect. □

We can now state the following theorem, first proved by Roberts in [6].

Theorem 1 If a graph G is weakly γ-perfect, then $\chi_n(G) = n\chi(G)$. ■

Example 6 Find $\chi_5(G)$ where G is the graph in Figure 3.

Solution: By Example 5 the graph in Figure 3 is weakly γ-perfect. As the solution to Example 5 shows, the chromatic number of G is four. Hence, by Theorem 1 it follows that $\chi_5(G) = 5 \cdot 4 = 20$. □

Graph Set Colorings

In the last section we considered the problem of assigning the same number of colors to each vertex of a graph. This was useful in modeling several types of applications. However, there are situations for which this generalization of graph coloring is not adequate. For example, in the channel assignment problem, each station may be assigned one or more channels, where not all stations are assigned the same number of channels. In scheduling examinations, each examination may be taken by students at several different time periods, with possibly different numbers of periods assigned to different examinations. In scheduling tasks, each task may require one or more time periods, where it is not necessarily the case that all tasks are assigned the same number of time periods. In maintenance scheduling some vehicles may require two or more repair bays, where not all vehicles require the same number of bays. (A large plane may use five bays, while a two-seater may use only one bay). To model such applications we need to make the following definition.

Definition 6 Let $G = (V, E)$ be a graph. A function S that assigns a set of colors to each vertex in V is called a **set coloring** if the sets $S(u)$ and $S(v)$ are disjoint whenever u and v are adjacent in G. □

Of course, an n-tuple coloring of a graph G is a special case of a set coloring in which every vertex is assigned the same number of colors. However, as the applications we have described show, it is often necessary to vary the number of colors assigned to different vertices.

Example 7 Construct a set coloring of C_5 where v_1, v_2, v_3, v_4, and v_5 are assigned 3 colors, 2 colors, 1 color, 3 colors, and 2 colors, respectively. (Here, the vertices of C_5 are as in Figure 2.)

Solution: We can construct such a coloring by assigning {red, green, blue} to v_1, {yellow, orange} to v_2, {red} to v_3, {green, blue, yellow} to v_4, and {orange, purple} to v_5. □

For different applications we need to minimize different parameters associated to a set coloring. For example, when scheduling final examinations, with each examination using one or more possible time periods, we might want to minimize the number of time periods used. This corresponds to minimizing the total number of different colors used in the set coloring of the associated graph. This quantity, the number of different colors used in the set coloring, is called the **order** of the set coloring. On the other hand, if we want to limit the number of hours required for proctoring, we would need to limit the sum of the number of time periods available for each examination. This corresponds to minimizing the sum of the sizes of the sets of colors used in the set coloring of the associated graph. This quantity, the sum of the sizes of the sets of colors, is called the **score** of the associated set coloring

Example 8 What is the order and score of the set coloring in Example 7?

Solution: Six different colors are used in this set coloring in Example 7. Hence the order of this set coloring is 6. Its score is the sum of the number of colors it assigns to each vertices, namely $3 + 2 + 1 + 3 + 2 = 11$. □

Graph T-Colorings

Consider the following channel assignment problem. Suppose that not only must interfering stations be assigned different channels, but also the separation between channels of interfering stations must not belong to a set of prohibited separations. For example, for the assignment of frequencies in ultra-high frequency (UHF) television, interfering stations cannot have the same frequency, nor can they have separations of 7, 14, or 15 channels. Suppose that in the maintenance of a fleet of planes, planes of a particular type cannot occupy repair bays that are adjacent because of interfering electrical signals. This leads us to the following definition.

Definition 7 Let $G = (V, E)$ be a graph and let T be a set of nonnegative integers containing 0. A function f from V to the set of positive integers is called a *T-coloring* if $|f(u) - f(v)| \notin T$ whenever u and v are adjacent vertices in G. □

Example 9 Construct a T-coloring of K_3 when $T = \{0, 1, 4\}$.

Solution: Let the vertices of K_3 be v_1, v_2, and v_3. We assign color 1 to v_1. Since the difference of the colors of adjacent vertices cannot be in T, we cannot assign any of colors 1, 2, or 5 to v_2. Suppose we assign color 3 to v_2. Then we cannot assign any of colors 1, 2, 3, 4, 5, or 7 to v_3. We assign color 6 to v_3. This gives a T-coloring of K_3.

Similarly, we could have assigned color 4 to v_1, color 6 to v_2, and color 1 to v_3. □

Example 10 Since ultra-high frequency (UHF) television stations that interfere cannot be assigned the same channel or channels differing by 7, 14, or 15 channels, to model the assignment of channels to UHF stations, we use a T-coloring where $T = \{0, 7, 14, 15\}$. □

There are several different useful measures of the "efficiency" of a T-coloring of a graph. Sometimes we may be concerned with the total number of colors used, while at other times we may be concerned with the range of colors used. For example, in the channel assignment problem we may be interested in how many channels are used, or, instead, we may be interested in the range of channels used (that is, the required bandwidth). By the **order** of a T-coloring we mean the number of colors used. By the **T-span** of a T-coloring we mean the maximum difference between colors, that is,

$$\max\{|f(u) - f(v)| \mid u, v \text{ are vertices of G}\}.$$

When we want to minimize the number of colors used, such as when we want to use the fewest possible channels, we need to find the **T-chromatic number** of G. When we want to find the smallest possible T-span of a T-coloring, such as when we want to find the smallest range of channels required, we need to find the **T-span** of G.

Minimizing the order and minimizing the T-span of a graph often requires different colorings, as the following example, adapted from Roberts [5], shows.

Example 11 Let $T = \{0, 1, 4, 5\}$. Find the minimum order and minimum span of a T-coloring of C_5.

Solution: The minimum order of a T-coloring of C_5 with $T = \{0, 1, 4, 5\}$ is the same as the chromatic number of C_5, namely 3, since we can assign color 1 to v_1 and v_3, color 4 to v_2 and v_4, and color 7 to v_5 (where we are using the assignment of vertices in Figure 2). The reader should verify that this T-coloring of C_5 with three colors has minimum span.

On the other hand, using five colors we can construct a T-coloring of C_5 with span equal to 4 by assigning color 1 to v_1, color 4 to v_2, color 2 to v_3, color 5 to v_4, and color 3 to v_5. It is impossible to find a T-coloring with span smaller than 4, as the reader should verify. Therefore, the T-span of C_5 is 4. □

The solution of Example 11 shows that we cannot necessarily minimize both the order and span of a T-coloring of a graph simultaneously.

The following theorem is useful in the computation of T-spans. It shows that to compute the T-span of a graph G we need only find its chromatic number k and then find the T-span of the complete graph with k vertices. (Since the proof is complicated, we will not give it here. A proof can be found in Cozzens and Roberts [1].)

Theorem 2 If G is a weakly γ-perfect graph and $\chi(G) = m$, then the T-span of G equals the T-span of K_m. ■

Example 12 Let $T = \{0, 1, 3, 4\}$. What is the T-span of the graph G in Figure 3?

Solution: By Example 5 the graph G is weakly γ-perfect and its chromatic number is 4. Hence, the T-span of G is the T-span of K_4. When $T = \{0, 1, 3, 4\}$ the T-span of K_4 is 9, since the minimum span of a coloring of K_4 is achieved by the assignment of colors 1, 3, 8, and 10 to the four vertices of K_4. (The details of this demonstration are left as Exercise 9.) □

Summary

We have seen how the study of applications of graph coloring to scheduling and assignment problems leads to the development of a variety of interesting generalizations of graph coloring. There are other generalization of graph coloring useful in constructing models which also have many interesting theoretical properties.

For example, in **list colorings** for vertices the colors that can be assigned to the vertices are restricted, in ***I*-colorings** an interval of real numbers is assigned to each vertex of a graph, and in ***J*-colorings** a union of intervals is assigned to each vertex of a graph. Also, to model some applications we need to combine two generalizations of graph colorings. For instance, we may need to assign two channels to broadcast stations so that interfering stations are not assigned the same channel or adjacent channels. This application requires the use of a combination of an n-tuple coloring and a T-coloring, which might be called an **n-tuple T-coloring**.

For more information and further references on these and other generalizations consult [5].

Suggested Readings

1. M. Cozzens and F. Roberts, "T-Colorings of Graphs and the Channel Assignment Problem," *Congressus Numerantium* , Vol. 35, 1982, pp. 191-208.

2. W. Hale, "Frequency Assignment: Theory and Applications," in *Proceedings of the IEEE*, Vol. 68, 1980, pp. 1497-1514.

3. R. Opsut and F. Roberts, "On the Fleet Maintenance, Mobile Radio Frequency, Task Assignment, and Traffic Phasing Problems," in *The Theory and Applications of Graphs*, Wiley, New York, 1981, pp. 479-492.

4. F. Roberts and B. Tesman, *Applied Combinatorics*, Second Edition, Prentice Hall, Upper Saddle River, N.J., 2005.

5. F. Roberts, "From Garbage to Rainbows: Generalizations of Graph Coloring and their Applications", *Proceedings of the Sixth International Conference on the Theory and Applications of Graphs* , Wiley, New York (to appear).

6. F. Roberts, "On the Mobile Radio Frequency Assignment Problem and the Traffic Light Phasing Problem", Annals of the New York Academy of Science, Vol. 319, 1979, pp. 466-83.

Exercises

1. Find $\chi_2(G)$ for: a) K_4 b) C_6 c) C_7 d) W_6 e) W_7 f) Q_3.

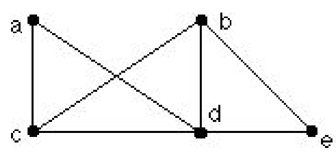
2. Assign three channels to each of five mobile telephone stations, using the smallest number of different channels possible, if Station 1 interferes with Stations 2, 4, and 5; Station 2 interferes with Stations 1, 3, and 5; Station 3 interferes with Stations 2 and 4; Station 4 interferes with Stations 1, 3, and 5; and Station 5 interferes with Stations 1, 2, and 4.

3. At the end of the year, final examinations, each with two parts, are to be scheduled. Examinations are given either in the morning or afternoon each day of a week. Schedule examinations using the smallest number of different time periods for the following courses: Graph Algorithms, Operating Systems, Number Theory, Combinatorics, Computer Security, Automata Theory, and Compiler Theory, if no courses with the same instructor or containing a common student can have a part scheduled at the same time. Assume that Professor Rosen teaches Number Theory and Computer Security, Professor Ralston teaches Graph Algorithms, Professor Carson teaches Operating Systems and Automata Theory, and Professor Bastian teaches Compiler Theory. Also assume that there are common students in Number Theory and Graph Algorithms, in Graph Algorithms and Operating Systems, in Automata Theory and Compiler Theory, in Computer Security and Automata Theory, Computer Theory and Graph Algorithms, and in Computer Security and Compiler Theory.

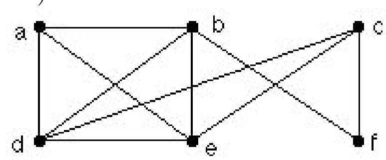
4. Show that if G is a bipartite graph with at least one edge and n is positive integer, then $\chi_n(G) = 2n$.

5. Find the clique number of each of the following graphs.

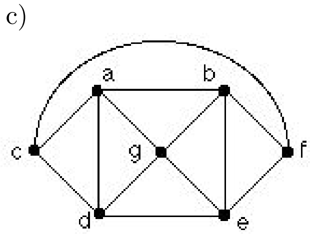
a)



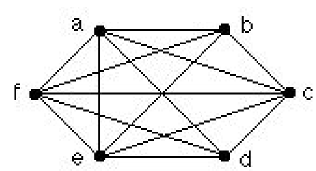
b)



c)



d)



6. Which of the graphs in Exercise 5 are weakly γ-perfect?

7. Show that every bipartite graph is weakly γ-perfect.

8. Find $\chi_2(G)$ for each graph G in Exercise 5.

9. Finish the argument in Example 12 to show that the T-span of K_4 is 9 when $T = \{0, 1, 3, 4\}$.

⋆10. Find the T-chromatic number and T-span of C_5 if T is

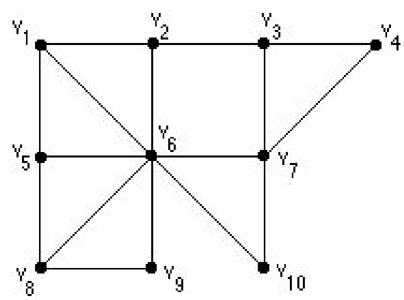
a) $\{0, 1\}$. b) $\{0, 1, 3\}$. c) $\{0, 1, 3, 4\}$.

⋆11. Find the T-chromatic number and T-span of K_4 for each of the sets T in Exercise 10.

⋆**12.** Find the T-span of each graph in Exercise 5, if $T = \{0, 1, 4\}$.

⋆**13.** We can use a greedy algorithm to construct an n-tuple coloring for a given graph G where n is a positive integer. First, we order the vertices of G as v_1, v_2,...,v_m and represent colors by positive integers. We assign colors 1,2,...,n to v_1. Then, once having assigned n colors to each of v_1, v_2,...,v_k, we assign the smallest numbered colors to v_{k+1} not already assigned to a vertex adjacent to v_{k+1}.

a) Use the greedy algorithm to construct a 2-tuple coloring of the following graph.



b) Show that this algorithm does not always construct an n-tuple coloring of minimal order.

14. Describe a greedy algorithm that constructs a set coloring of a given graph G where each vertex v is to be assigned a set $S(v)$ of colors where the size of $S(v)$ is specified for each vertex.

⋆**15.** We can use a greedy algorithm to construct a T-coloring for a given graph G and set T. First, we order the vertices of G as v_1, v_2,...,v_n and represent colors by positive integers. We assign color 1 to v_1. Once we have assigned colors to v_1,v_2,...,v_k, we assign the smallest numbered color to v_{k+1} so that the separation between the color of v_{k+1} and the colors of vertices adjacent to v_{k+1} that are already colored is not in T.

a) Use this algorithm to construct a T-coloring of the graph G in Exercise 13 where $T = \{0, 1, 4, 5\}$.

b) Show that this algorithm does not always construct a T-coloring of minimal span.

16. Describe two different problems that can be modeled using list colorings.

17. Describe two different problems that can be modeled using I-colorings.

18. Describe two different problems that can be modeled using J-colorings.

Computer Projects

1. Given the adjacency matrix of a graph G and a positive integer n, construct an n-tuple coloring of G.
2. Given the adjacency matrix of a graph G and a set of nonnegative integers T containing 0, construct a T-coloring of G.