Rosen, Discrete Mathematics and Its Applications, 7th edition
Extra Examples
Section 1.8—Proof Methods and Strategy

---

p.93, icon at Example 1
#1. Prove that there is only one pair of positive integers that is a solution to $3x^2 + 2y^2 = 30$.

Solution:
The two terms on the left side of the equation each involve a square, and hence the sum exceeds 30 for small values of x and y. In particular, in order to have a solution, we must have $x \leq 3$ and $y \leq 4$. The number of such pairs is sufficiently small that we can use an exhaustive proof, considering the cases $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$. The only solution among these twelve pairs is $(2,3)$. That is, the only integer solution to $3x^2 + 2y^2 = 30$ is $x = 2$, $y = 3$.

---

p.93, icon at Example 3
#1. Prove that the square of every even integer ends in 0, 4, or 6.

Solution:
Every even integer n can be written as $n = 10k + r$ where $r = 0, 2, 4, 6, 8$. (For example, $34 = 10 \cdot 3 + 4$ and $6 = 6 \cdot 0 + 6$.) Examine each of these five cases separately:

$n = 10k + 0$: $n^2 = 100k^2$, which ends in 0 (it is a multiple of 10)
$n = 10k + 2$: $n^2 = 100k^2 + 40k + 4 = 10(10k^2 + 4k) + 4$ and hence ends in 4
$n = 10k + 4$: $n^2 = 100k^2 + 80k + 16 = 10(10k^2 + 8k + 1) + 6$ and hence ends in 6
$n = 10k + 6$: $n^2 = 100k^2 + 120k + 36 = 10(10k^2 + 12k + 3) + 6$ and hence ends in 6
$n = 10k + 8$: $n^2 = 100k^2 + 160k + 64 = 10(10k^2 + 16k + 6) + 4$ and hence ends in 4.

---

p.93, icon at Example 3
#2. Prove that the following is true for all real numbers x and y: $\max(x, y) = \frac{1}{2}(x + y + |x - y|)$.

Solution:
We will carry out a proof by cases. Because the absolute value function is used in the given equation, and because the absolute value of a number depends on whether the number is negative or nonnegative, it would be reasonable to consider two cases: $x - y < 0$ and $x - y \geq 0$. That is, we consider $x < y$ and $x \geq y$.

Case 1: $x < y$. In this case,
$$\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y - (x - y)) = \frac{1}{2}(x + y + y - x) = \frac{1}{2}(2y) = y.$$
But if $x < y$, then we also have $\max(x, y) = y$.

Case 2: $x \geq y$. In this case
$$\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + (x - y)) = \frac{1}{2}(x + y + x - y) = \frac{1}{2}(2x) = x.$$

But if $x \geq y$, then we also have $\max(x, y) = x$.

Thus, in both cases we have $\max(x, y) = \frac{1}{2}(x + y + |x - y|)$.

---

p.93, icon at Example 3
#3. Prove that the square of every odd integer ends in 1, 5, or 9.

Solution:
Every odd integer n can be written as $n = 10k + r$ where $r = 1, 3, 5, 7, 9$. (For example, $163 = 10 \cdot 16 + 3$ and $7 = 10 \cdot 0 + 7$.) Examine each of these five cases separately, writing each square as "multiple of 10, plus an integer x". The integer x will be the units' digit.

$n = 10k + 1$: $n^2 = 100k^2 + 20k + 1 = 10(10k^2 + 2k) + 1$, which ends in 1.

$n = 10k + 3$: $n^2 = 100k^2 + 60k + 9 = 10(10k^2 + 6k) + 9$, which ends in 9.

$n = 10k + 5$: $n^2 = 100k^2 + 100k + 25 = 10(10k^2 + 10k + 2) + 5$, which ends in 5.

$n = 10k + 7$: $n^2 = 100k^2 + 140k + 49 = 10(10k^2 + 14k + 4) + 9$, which ends in 9.

$n = 10k + 9$: $n^2 = 100k^2 + 180k + 81 = 10(10k^2 + 18k + 8) + 1$, which ends in 1.

Therefore, in each of the five possible cases the square ends in 1, 5, or 9.

---

p.96, icon at Example 10
#1. Prove that there are numbers x and y whose sum is 5 and whose product is 2. (Note that we are only required to show that x and y exist; we are not required to find specific values for x and y.)

Solution:
We need to prove that x and y exist such that $x + y = 5$ and $xy = 2$. To find x and y, rewrite the first equation as $y = 5 - x$ and substitute for y in the second equation: $x(5 - x) = 2$. This yields the quadratic equation $x^2 - 5x + 2 = 0$. We could solve the equation (by using the quadratic formula) to find a specific value for x, which would then yield a specific value for y, but we do not need to do this. We are only asked to prove the *existence* of x and y.

We only need to use the discriminant, $b^2 - 4ac$, to see if there are any real number solutions. We obtain $b^2 - 4ac = (-5)^2 - 4(1)(2) = 17$. Because the discriminant is greater than 0, the given equation has two real numbers as solutions. This guarantees that there is such an x. Because such a number x exists, so does the corresponding number y, which is equal to $5 - x$.

[Note: If we wish to produce specific numbers x and y that satisfy $x + y = 5$ and $xy = 2$, we use the quadratic formula to obtain $x = \frac{5 + \sqrt{17}}{2}$ and $y = \frac{5 - \sqrt{17}}{2}$ as two numbers whose sum is 5 and whose product is 2.]

---

p.99, icon at Example 13
#1. Show that if x is a nonzero rational number, then there is a unique rational number y such that $xy = 2$.

Solution:
Note that if x is a nonzero rational number, there exists integers a and b such that $x = a/b$ and $b \neq 0$. It follows that if $y = 2b/a$, then $xy = (a/b)(2b/a) = 2$. Because $y = 2b/a$ is a rational number, it follows that for every nonzero integer x, there exists a rational number y such that $xy = 2$. This completes the existence part of the proof.

Now suppose given the nonzero rational number x, z is a rational number with $xz = 2$. It follows that $xz = xy$ where $y = 2b/a$. Because $x \neq 0$, we can divide both sides of the equation $xz = xy$ by x to find that $z = y$. This completes the uniqueness part of the proof.

---

p.100, icon at Example 14
#1. Prove that the square of every odd integer has the form $8k + 1$, where k is an integer.

Solution:
We begin by taking an odd integer n, which must have the form $n = 2i + 1$ for some integer i. Then $n^2 = (2i + 1)^2 = 4i^2 + 4i + 1$. We need to show that this has the form $8k + 1$. Using backward reasoning, this will follow if we can show that $4i^2 + 4i$ can be written as $8k$.

But $4i^2 + 4i = 4i(i + 1)$. Note that $i(i + 1)$ is the product of two consecutive integers. Because every other integer is even, either i or $i + 1$ is even. Therefore the product $i(i + 1)$ is even, and hence can be written as $2j$ for some integer j.

Therefore
$$4i^2 + 4i = 4(i(i + 1)) = 4(2j) = 8j.$$
Because we can write $4i^2 + 4i = 8j$, it follows that

$$n^2 = 4i^2 + 4i + 1 = (4i^2 + 4i) + 1 = 8j + 1.$$

---

p.100, icon at Example 14
#2. Suppose a and b are integers such that $2a = b^2 + 3$. Prove that a is the sum of three squares.

Solution:
We first observe that $b^2 + 3$ is even — it is equal to $2a$. Because 3 is odd, b^2 must also be odd. Therefore b must be odd and we can write $b = 2n + 1$. Therefore

$$2a = b^2 + 3 = (2n + 1)^2 + 3 = 4n^2 + 4n + 4.$$

Dividing by 2, we have $a = 2n^2 + 2n + 2$. We need to examine $2n^2 + 2n + 2$ and try to write it as the sum of three squares.

If we write $2n^2$ as $n^2 + n^2$, we have

$$a = n^2 + n^2 + 2n + 2.$$

This gives two squares, but the remaining terms, $2n + 2$, does not appear to be a square. But if we write this as $a = n^2 + n^2 + 2n + 1 + 1$ we obtain a as sum of three squares:

$$a = n^2 + (n^2 + 2n + 1) + 1 = n^2 + (n + 1)^2 + 1.$$

3

p.101, icon at Example 16

#1. Over the centuries, mathematicians have tried to adapt the proofs of others to obtain new results. A classic example of this is the proof of the Four Color Theorem. The Four Color Theorem states that the countries on every map can be colored with at most four colors so that two countries that share a common border have different colors.

In the nineteenth century it was proved that five colors are sufficient to color the countries on any map so that countries that share a common border receive different colors. No one was able to produce a map that required five colors, but no one was able to prove that four colors were sufficient to color every possible map. In 1976, two mathematicians, Kenneth Appel and Wolfgang Haken, were able to prove that four colors suffice to color the countries of every map so no countries that share a common border have the same color. This proof, complicated and very lengthy, was an adaptation of the much simpler proof that five colors always suffice. Map coloring problems will be discussed in detail in Section 9.8.

---

p.101, icon at Example 16

#2. (Adapted from Problem A4 from the 1988 William Lowell Putnam Mathematics Competition.)

(a) Suppose that every point of the plane is painted one of two colors, a or b. Must there be two points of the same color that are exactly one inch apart?

(b) Suppose that every point of the plane is painted one of three colors, a, b, or c. Must there be two points of the same color that are exactly one inch apart?

(c) Prove that if nine colors are allowed, a coloring is possible with the property that no two points one inch apart have the same color.

Solution:

(a) Our strategy here is to examine three points and conclude that two of the three must have the same color. To do this, take an equilateral triangle with each of its sides one inch long. Of the three vertices of the triangle, at least two (say P and Q) must have the same color because there are only two possible colors. Therefore P and Q have the same color and are 1 inch apart.

(b) Note that we cannot adapt the proof from part (a) to find four points, each one inch away from the other three, because it is impossible to obtain such a set of four points in the plane. (You should show that such a set does not exist.)

To show that there must be two points of the same color one inch apart, we will give a proof by contradiction. Suppose it is possible to color every point of the plane such that no two points that are one inch apart have the same color. We begin as in part (a) by taking an equilateral triangle one inch long on each side. The colors of the three vertices of the triangle must all be different — color the three points a, b, and c. Fix the point colored a (call the point A) and consider all equilateral triangles of side length one with one vertex at A. The other two vertices of such triangles trace a circle S of radius one with center at A, as in the following figure. Note that each point on S must be colored b or c. (If any point of S is colored a, then that point and the center of the circle S have the same color and are one inch apart, which would be a contradiction.)

Take any one triangle with a vertex at A that has its other vertices on the circle S — the other two vertices of the triangle are colored b and c (call these points B and C). Flip this triangle over the side BC to obtain a new point, which must also be colored a. (Otherwise we would have two points one inch apart both colored b or both colored c.) This is illustrated for one triangle in the figure below.

4
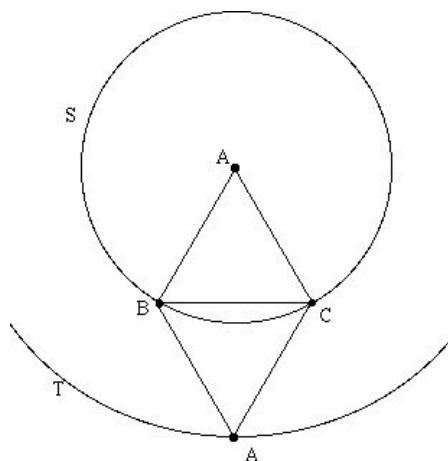
If this is done for all triangles with a vertex at A and sides one inch long, the new vertex of each flipped triangle must also be colored a. This gives a circle T with the property that all points on the circle must be colored a. Take any two points on this circle T that are one inch apart (which is possible because its diameter is at least one), and a contradiction is obtained.

(c) One such coloring can be obtained by using the following figure, where each square has diagonal length 0.9 inches. Color all points in the same square the same color. Use this 3×3 square as a "tile", and tile the plane with such squares, arranging the squares left-to-right and bottom-to-top. No two points in the same square of a tile can have the same color (the squares are not large enough). Also, squares of the same color in different tiles are too far apart to have points one inch apart.

A	B	C
D	E	F
G	H	I

---

p. 102, icon at Example 17

1. Find a counterexample to the statement that the sum of two irrational numbers is also an irrational number.

Solution.

To find a counterexample, we will take the sum of $\sqrt{2}$ and $-\sqrt{2}$, which are both irrational numbers. The sum of these numbers is $\sqrt{2} + (-\sqrt{2}) = 0$, which is rational. Note that we could have taken any irrational number and its negative to find a counterexample.

5

p. 102, icon BELOW Example 17

1. By examining the small powers of 2 and of 3, what conjectures can you make about how close a power of 2 can be to a power of 3?

Solution.

The smallest powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ... and the smallest powers of 3 are 3, 9, 27, 81, 243, 729, 2187,... . We first note that none of these powers are ever the same, so we can quickly conjecture that the two sequences of positive powers have no common terms. Using material from Chapter 4, we will see that this conjecture is true.

Looking at these powers we see that 2 and 3 differ by 1, as do $2^3 = 8$ and $3^2 = 9$, but no other pairs of powers differ by 1 in our list. We can get more evidence for the conjecture that the only powers of 2 and 3 that differ by one are these two pairs by extending our lists of powers further. More formally, this leads to the conjecture that the only solutions to the equation $2^a - 3^b = \pm 1$ where a and b are positive integers are when $a = b = 1$ and when $a = 3$ and $b = 2$. This conjecture turns out to be true and can be proved using methods based on the contents of Chapter 4.

We leave it to the student to make other conjectures about possible difference of powers of 2 and of 3.

this conjecture:

$$xy + yz + zx + 1 = \underbrace{(a-1)}_{x} \cdot \underbrace{(b-1)}_{y} + \underbrace{(b-1)}_{y} \cdot \underbrace{(1)}_{z} + \underbrace{(1)}_{z} \cdot \underbrace{(a-1)}_{x} + 1$$

$$= ab - a - b + 1 + b - 1 + a - 1 + 1 = ab = n.$$

Therefore, if n is a composite integer ab (where $a > 1$ and $b > 1$), we can write $n = xy + yz + zx + 1$ as $n = (a-1)(b-1) + (b-1) \cdot 1 + 1 \cdot (a-1) + 1$.

---

p.103, icon at Example 19
#1. A rectangular floor is tiled using two kinds of tiles, square 2×2 tiles and rectangular 1×4 tiles. Suppose that one tile is destroyed, but that one tile of the other kind is available. It is possible to tile the entire floor, using the original tiles with this one replacement, by rearranging the tiles?

Solution:
The answer is no. To see this, suppose that we color the rectangular floor so that all squares of the odd numbered rows are colored white while squares in the even numbered rows begin with a black square and are alternately colored white and black. With this coloring a 2×2 square always covers exactly one black square while a 1×4 tile covers either zero or two black squares. Consequently, it is impossible to tile the same rectanglar floor after exchanging one tile for one of the other kind and rearranging because the tile replacement changes the parity of the number of black squares covered.

---

p.103, icon at Example 19
#2. Can you tile a 17×28 checkerboard using 4×7 tiles?

Solution:
Even though the number of squares on a 17×28 checkerboard is divisible by the number of squares of a 4×7 tile, the answer is no. To see this, note that the first column of 17 squares must be covered by a combination of 4×7 tiles placed either vertically or horizontally. This means that 17 must be the sum of a multiple of 4 and a multiple of 7. However, by exhaustion, we can see that there is no solution in nonnegative integers a and b of the equation $17 = 4a + 7b$. Hence, such a tiling is impossible.

---

p.103, icon at Example 19
#3. An *T-tetromino* consists of a row of three squares with a fourth square directly above the middle square. Show that a 14×14 checkerboard cannot be tiled with T-tetrominoes.

Solution:
Suppose that the squares of the 14×14 checkerboard are colored white and blank in the usual alternating way. There are $14 \cdot 14 = 196$ squares on this checkerboard; 98 are black and 98 are white. Because each tetromino covers four squares, a total of $196/4 = 49$ tetrominoes are needed to tile the checkerboard. Note that each T-tetromino covers either three white and one black square or one white and three black squares. If there are k T-tetrominoes each covering three white and one black square, there are $49 - k$ T-tetrominoes

each covering one white and three black squares. It follows that the number of white squares covered equals $3k + (49 - k) = 2k + 49$. Because there are 98 white squares, we have $96 = 2k + 49$. But this is impossible because 96 and $2k$ are even, but 49 is odd.

---

p.103, icon at Example 19
#4. Show that in any tiling of an 8×8 checkerboard by tetrominoes, where any of the five different kinds of tetrominoes can be used, the number of T-tetrominos must be even.

Solution:
Suppose that the squares of the 8×8 checkerboard are colored in the usual alternating way, with 32 white squares and 32 black squares. Note that except for the T-tetromino, each of the other four kinds of tetrominoes covers two white and two black squares. However, a T-tetromino covers either one or three black squares. Consequently, if the tiling contains an odd number of T-tetrominoes, it covers an odd number of black squares. Because there are 32 black squares in this checkerboard, this is a contradiction. It follows that the number of T-tetrominoes must be even.

---

p.103, icon at Example 19
#5. In a tiling of a checkerboard by dominoes, a *fault line* is a vertical or horizontal line that cuts the checkerboard into two pieces without passing through any of the dominoes. Show that whenever a 6×6 checkerboard is tiled with dominoes, the tiling has a fault line. That is, no matter how the 6×6 checkerboard is tiled with dominoes, it is possible to cut the checkerboard in two without passing through one of the dominoes.

Solution:
We will use a proof by contradiction. Suppose that there was a tiling of the 6×6 checkerboard that did not have a fault line. In particular, each of the five vertical lines, beginning at the right of each of the first five columns of squares, which are potential fault lines, and each of the five horizontal lines, beginning at the bottom of each of the first five rows of squares, which are potential fault lines, must cross a domino. Consider each of these five vertical lines. Note that there are an even number of squares to the left of this line. Consequently, this vertical line must cross at least two dominoes, for if it just crossed one, this would mean that there were an odd number of squares to its left. Similarly, each of the five horizontal lines which are potential fault lines must cross at least two dominoes in the tiling. Because each of these ten lines (the five vertical lines and the five horizontal lines, each of which is a potential fault line) must cross at least two dominoes, there must be at least $10 \times 2 = 20$ dominoes in the tiling. However, a tiling of a 6×6 checkerboard only contains 18 dominoes. Therefore, every tiling of a 6×6 checkerboard by dominoes must have a fault line.

---

p.103, icon at Example 19
#6. Prove or disprove that there is a tiling of the 5×6 checkerboard that does not have a fault line, that is, a vertical or horizontal line that cuts the checkerboard in two without passing through one of the dominoes.

Solution:

It is possible to tile the 5×6 checkerboard so that there is no fault line for the tiling. To describe such a tiling, suppose that we number the squares of the checkerboard using the numbers 1 through 5 for the tiles in the first row, going left to right, the numbers 6 through 10 for the tiles in the second row, going left to right, and so on. When the dominoes in the tiling cover squares 1-6,2-3,4-5,7-8,9-14,10-15,11-12,13-18,16-21,17-22,19-20,23-24,25-30,26-27, and 28-29, the resulting tiling does not have a fault line (and is said to be *fault-free*. We leave the verification that each of the eight possible fault lines passes through one of these dominoes.

---

p.103, icon at Example 19
#7. How many different pentominoes, that is, arrangements of five squares of a checkerboard joined along edges, are there, where two such arrangements are considered the same if one can be obtained from the other by a rotation or a flipping?

Solution:
There are 12 different pentominoes. We can describe them by indicating which squares are part of the pentomino in the smallest rectangle on the checkerboard in which they fit. We use a 1 to indicate the presence of a square to indicate its absence: 1,1,1,1,1 (straight pentomino, resembling the letter I); 10,10,10,11 (resembles the letter L); 01,11,10,10; 01,11,01,01; 100,100,111; 111,010,010; 001,011,110; 010,111,010; 101,111; 110,010,011; 011,110,010; 11,11,10. We can prove, using a proof by exhaustion, that there are only 12 different pentominoes by considering all possible ways to construct a pentomino and show that we can rotate or flip it to obtain one of these 12.