

Rosen, Discrete Mathematics and Its Applications, 7th edition
Extra Examples
Section 4.1—Divisibility and Modular Arithmetic



— Page references correspond to locations of Extra Examples icons in the textbook.

p.238, icon at Example 2

#1. Certain rules allow us to determine by inspection when a positive integer n is divisible by a positive integer k . For example, $5 \mid n$ if and only if n ends in the digit 5 or 0. Similarly, $2 \mid n$ if and only if n ends in one of the digits 0, 2, 4, 6, 8.

There is also a rule to determine divisibility by 3:

$$3 \mid n \text{ if and only if the sum of the digits in } n \text{ is divisible by 3.}$$

For example $3 \mid 478,125$ because the sum of the six digits in 478,125 is 27, which is divisible by 3. Why does the rule work?

Solution:

We want to show that if n is written as the string of digits $a_k a_{k-1} \dots a_2 a_1 a_0$, then $3 \mid n$ if and only if $3 \mid (a_k + a_{k-1} + \dots + a_2 + a_1 + a_0)$.

Begin by writing

$$\begin{aligned} n &= a_k 10^k + a_{k-1} 10^{k-1} + \dots + a_2 10^2 + a_1 10 + a_0 \\ &= [(10^k - 1) + 1]a_k + [(10^{k-1} - 1) + 1]a_{k-1} + \dots + [(10^2 - 1) + 1]a_2 + [(10 - 1) + 1]a_1 + a_0 \\ &= \underbrace{[(10^k - 1)a_k + (10^{k-1} - 1)a_{k-1} + \dots + (10^2 - 1)a_2 + (10 - 1)a_1]}_{=A} + \underbrace{[a_k + a_{k-1} + \dots + a_2 + a_1 + a_0]}_{=B}. \end{aligned}$$

Note that in A each summand is of the form $(10^i - 1)a_i$ and is divisible by 3 because each coefficient is a power of 10, with 1 subtracted, that is, each summand is a string of 9's. Therefore A is divisible by 3 because A is the sum of numbers of this form.

Now suppose that $3 \mid n$. Write $n - A = B$. The numbers n and A are both divisible by 3. Therefore B is divisible by 3 because B is the difference of two numbers divisible by 3. But B is the sum of the digits in n . Therefore, if $3 \mid n$, then the sum of the digits of n is divisible by 3.

Conversely, suppose 3 divides the sum of the digits in n . Then $3 \mid (a_k + a_{k-1} + \dots + a_2 + a_1 + a_0)$, which is B . But we already know that $3 \mid A$. Therefore 3 divides their sum $A + B$, which is n .

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#2.

- (a) Find the number of positive integer divisors of $648 = 2^3 3^4$.
- (b) Find the sum of all positive integer divisors of 648.

Solution:

- (a) Each divisor must have the form $2^i 3^j$ where $0 \leq i \leq 3$ and $0 \leq j \leq 4$. Hence, there are $4 \cdot 5 = 20$ divisors of 648.
- (b) The sum of the divisors is

$$\begin{aligned}
&2^03^0 + 2^03^1 + 2^03^2 + 2^03^3 + 2^03^4 + \\
&\quad 2^13^0 + 2^13^1 + 2^13^2 + 2^13^3 + 2^13^4 + \\
&\quad\quad 2^23^0 + 2^23^1 + 2^23^2 + 2^23^3 + 2^23^4 + \\
&\quad\quad\quad 2^33^0 + 2^33^1 + 2^33^2 + 2^33^3 + 2^33^4 = \\
&2^0(3^0 + 3^1 + 3^2 + 3^3 + 3^4) + \\
&\quad 2^1(3^0 + 3^1 + 3^2 + 3^3 + 3^4) + \\
&\quad\quad 2^2(3^0 + 3^1 + 3^2 + 3^3 + 3^4) + \\
&\quad\quad\quad 2^3(3^0 + 3^1 + 3^2 + 3^3 + 3^4) = \\
&2^0 \cdot 121 + 2^1 \cdot 121 + 2^2 \cdot 121 + 2^3 \cdot 121 = 15 \cdot 121 = 1815.
\end{aligned}$$

If you are familiar with sigma notation (covered in Section 2.4), this summation process can be more compactly written as

$$\begin{aligned}
\sum_{i=0}^3 \sum_{j=0}^4 2^i 3^j &= 2^0 \sum_{j=0}^4 3^j + 2^1 \sum_{j=0}^4 3^j + 2^2 \sum_{j=0}^4 3^j + 2^3 \sum_{j=0}^4 3^j \\
&= (2^0 + 2^1 + 2^2 + 2^3)121 \\
&= 1815.
\end{aligned}$$

p.240, icon at Example 4

#1. For each pair of numbers, when the division algorithm is used to divide a by d , what are the quotient q and remainder r ?

- (a) $a = 88, d = 11.$
- (b) $a = -29, d = 9$
- (c) $a = 58^{237}, d = 58^{168}$

Solution:

- (a) Because $88 = 11 \cdot 8 + 0$, we have $q = 8, r = 0$. (The fact that $r = 0$ says that $11|88$.)
- (b) Because $-29 = 9 \cdot (-4) + 7$, we have $q = -4$ and $r = 7$. (Note that although we can write $-29 = 9 \cdot (-3) + (-2)$, we cannot use -2 as r because r is not allowed to be negative.)
- (c) We do not need to perform the exponentiations to find a and d . We need only observe that a is a multiple of d : $58^{237} = 58^{168} \cdot 58^{69}$ (recall the rule for exponents: $a^b a^c = a^{b+c}$). Therefore $58^{237} = 58^{168} \cdot 58^{69} + 0$ and we have $q = 58^{69}$ and $r = 0$.