Extra Examples — *Page references correspond to locations of Extra Examples icons in the textbook.*

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p.316, icon at Example 1

#1. Use the Principle of Mathematical Induction to prove that

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^n = \frac{5^{n+1} - 1}{4} \quad \text{for all } n \geq 0.$$

Solution:

Let $P(n)$ be $1 + 5 + 5^2 + 5^3 + \cdots + 5^n = \dfrac{5^{n+1} - 1}{4}$.

BASIS STEP: $P(0)$: $1 = \dfrac{5^{0+1} - 1}{4}$. (Note that the sum on the left side of $P(0)$ begins and ends with 5^0, and hence is just the first term, 1.) $P(0)$ is true because both sides equal 1.

INDUCTIVE STEP: $P(k) \to P(k+1)$: Suppose for some k, $P(k)$ is true; i.e.

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^k = \frac{5^{k+1} - 1}{4}.$$

We need to show that the next statement, $P(k+1)$, is true:

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^{k+1} = \frac{5^{k+2} - 1}{4}$$

To do this, begin with $P(k)$ and add the next term, 5^{k+1}, to both sides. Then show that this is $P(k+1)$.

$$
\begin{aligned}
1 + 5 + 5^2 + 5^3 + \cdots + 5^k &= \frac{5^{k+1} - 1}{4} \\
+5^{k+1} &\quad +5^{k+1} \\
\hline
1 + 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} &= \frac{5^{k+1} - 1 + 4 \cdot 5^{k+1}}{4} \\
1 + 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} &= \frac{(1+4)5^{k+1} - 1}{4} \\
1 + 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} &= \frac{5 \cdot 5^{k+1} - 1}{4} \\
1 + 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} &= \frac{5^{k+2} - 1}{4}.
\end{aligned}
$$

This is $P(k+1)$. Hence $P(k) \to P(k+1)$ is true.

Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 0$.

Note: Alternately, the proof of $P(k) \to P(k+1)$ can be written in this form. Note that the assumption that $P(k)$ is true is being used in the substitution in the first step.

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} = (1 + 5 + 5^2 + 5^3 + \cdots + 5^k) + 5^{k+1}$$
$$= \frac{5^{k+1} - 1}{4} + 5^{k+1}$$
$$= \frac{5^{k+1} - 1 + 4 \cdot 5^{k+1}}{4}$$
$$= \frac{(1 + 4)5^{k+1} - 1}{4}$$
$$= \frac{5 \cdot 5^{k+1} - 1}{4}$$
$$= \frac{5^{k+2} - 1}{4}.$$

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p.316, icon at Example 1

#2. Use the Principle of Mathematical Induction to prove the "generalized" distributive law

$$a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots + ab_n$$

for all integers $n \geq 2$.

Solution:

For each positive integer n, let $P(n)$ be the statement $a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots + ab_n$. To give a proof using the Principle of Mathematical Induction, we need to show that $P(2)$ is true and that $P(k) \rightarrow P(k+1)$ is true for all $k \geq 2$.

$P(2)$ is true: $P(2)$ states that $a(b_1 + b_2) = ab_1 + ab_2$, which is the usual distributive law.

$P(k) \rightarrow P(k+1)$ is true: Suppose $k \geq 2$ and $a(b_1 + b_2 + \cdots + b_k) = ab_1 + ab_2 + \cdots + ab_k$ is true. We need to show that $a(b_1 + b_2 + \cdots + b_{k+1}) = ab_1 + ab_2 + \cdots + ab_{k+1}$. The key to showing this it rewriting the sum on the left side as the sum of two numbers: $b_1 + b_2 + \cdots + b_{k+1} = (b_1 + b_2 + \cdots + b_k) + b_{k+1}$. We can then use $P(2)$ to split this apart. We have

$$a(b_1 + b_2 + \cdots + b_{k+1}) = a((b_1 + b_2 + \cdots + b_k) + b_{k+1})$$
$$= a(b_1 + b_2 + \cdots + b_k) + ab_{k+1}$$
$$= (ab_1 + ab_2 + \cdots + ab_k) + ab_{k+1}$$
$$= ab_1 + ab_2 + \cdots + ab_k + ab_{k+1}.$$

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p.316, icon at Example 1

#3. Use the Principle of Mathematical Induction to prove that

$$\sum_{i=1}^{n} (2i + 3) = n(n + 4) \quad \text{for all } n \geq 1.$$

Solution:

2

Let $P(n)$ be $\displaystyle\sum_{i=1}^{n}(2i+3) = n(n+4)$.

BASIS STEP: $P(1)$ is true because $2 + 3 = 1(1 + 4)$.

INDUCTIVE STEP: $P(k) \to P(k+1)$: Suppose $P(k)$ is true; that is, $\displaystyle\sum_{i=1}^{k}(2i+3) = k(k+4)$.

Therefore,

$$\sum_{i=1}^{k+1}(2i+3) = k(k+4) + (2k+5)$$
$$= k^2 + 6k + 5$$
$$= (k+1)(k+5)$$

which is $P(k+1)$. Thus, $P(k) \to P(k+1)$ is true.

Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

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p.316, icon at Example 1
#4. Find a formula for

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)$$

for $n \geq 2$, and use the Principle of Mathematical Induction to prove that the formula is correct.

Solution:
We first need to guess at a formula for the product. Using $n = 2, 3, 4, 5$ yields:

$$\left(1 - \frac{1}{2^2}\right) = \frac{3}{4}$$

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) = \frac{3}{4} \cdot \frac{8}{9} = \frac{2}{3}$$

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} = \frac{5}{8}$$

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right) = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \cdot \frac{24}{25} = \frac{3}{5}.$$

Thus, the products are $\dfrac{3}{4}, \dfrac{2}{3}, \dfrac{5}{8}, \dfrac{3}{5}$. Searching for a pattern, we see that we can rewrite these fractions as $\dfrac{3}{4}, \dfrac{4}{6}, \dfrac{5}{8}, \dfrac{6}{10}$.

This suggests $\dfrac{n+1}{2n}$ as the general form for the sum.

Let $P(n)$ be

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

We now try to show that this is true for all $n \geq 2$.

3

BASIS STEP: $P(2)$ is true: $P(2)$ states that $\left(1 - \dfrac{1}{2^2}\right) = \dfrac{2+1}{2 \cdot 2}$, which is true because both sides of this equation are equal to $\dfrac{3}{4}$.

INDUCTIVE STEP: $P(k) \to P(k+1)$: Suppose $P(k)$ is true for an integer $k \geq 2$.

Therefore $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\left(1 - \dfrac{1}{4^2}\right) \cdots \left(1 - \dfrac{1}{k^2}\right) = \dfrac{k+1}{2k}$.

Multiply both sides of this equation by $1 - \dfrac{1}{(k+1)^2}$ to obtain

$$
\begin{aligned}
\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) &= \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\left(\frac{(k+1)^2 - 1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k} \cdot \frac{k(k+2)}{(k+1)^2} \\
&= \frac{k+2}{2(k+1)},
\end{aligned}
$$

which is $P(k+1)$.

We have shown that $P(2)$ is true and that $P(k) \to P(k+1)$ is true for all $k \geq 2$. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 2$.

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p.319, icon at Example 5

#1. Use the Principle of Mathematical Induction to show that the following inequality is true for all integers $n \geq 2$:

$$
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}.
$$

Solution:

Let $P(k)$ be the inequality $\displaystyle\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$. We need to show that $P(2)$ is true and that $P(k) \to P(k+1)$ is true for all $k \geq 2$.

$P(2)$ is true: $P(2)$ states that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$. This is true because the left side is greater than 1.7, while the right side is less than 1.5.

$P(k) \to P(k+1)$ is true: Assume k is a positive integer $(k \geq 2)$ such that $\displaystyle\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$ is true. We need to show that $\displaystyle\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$ is true. Take the inequality $P(k)$ and add $\dfrac{1}{\sqrt{k+1}}$ to both sides, obtaining

$$
\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}.
$$

If we can show that the right side, $\sqrt{k} + \dfrac{1}{\sqrt{k+1}}$, is greater than $\sqrt{k+1}$, we will have proved that $P(k+1)$ is true. We will use backward reasoning.

4

Begin by multiplying both sides of $\sqrt{k} + \dfrac{1}{\sqrt{k+1}} > \sqrt{k+1}$ by the positive number $\sqrt{k+1}$ to obtain the equivalent inequality

$$\sqrt{k+1}\left(\sqrt{k} + \frac{1}{\sqrt{k+1}}\right) > \sqrt{k+1}\left(\sqrt{k+1}\right),$$

which simplifies to

$$\sqrt{k}\sqrt{k+1} + 1 > k+1,$$

or

$$\sqrt{k(k+1)} > k.$$

But this last inequality is true because $\sqrt{k(k+1)} > \sqrt{k \cdot k} = k$. Reversing the steps shows that $P(k+1)$ is true.

Because both $P(2)$ and $P(k) \to P(k+1)$ are true, the Principle of Mathematical Induction guarantees that $P(n)$ is true for all $n \geq 2$.

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p.319, icon at Example 5
#2. Use the Principle of Mathematical Induction to prove that $n^2 - 5n + 3 > 0$ for all $n \geq 5$.

Solution:
Let $P(n)$ be $n^2 - 5n + 3 > 0$.

BASIS STEP: The basis step is $P(5)$. $P(5)$ states that $5^2 - 5 \cdot 5 + 3 > 0$, which is true because $3 > 0$.

INDUCTIVE STEP: $P(k) \to P(k+1)$: Assuming $k^2 - 5k + 3 > 0$, we need to show that $(k+1)^2 - 5(k+1) + 3 > 0$ is true. But

$$(k+1)^2 - 5(k+1) + 3 = k^2 + 2k + 1 - 5k - 5 + 3 = (k^2 - 5k + 3) + (2k - 4).$$

We know that $k^2 - 5k + 3$ is positive (it is $P(k)$), and $2k - 4$ is positive because k is at least 5. Hence the sum is positive; that is, $P(k+1)$ is true.

We have shown that $P(5)$ is true and that $P(k) \to P(k+1)$ is true for all $k \geq 5$. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 5$.

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p.321, icon at Example 8
#1. Prove that 6 is a divisor of $4^n + 7^n + 1$ for all positive integers n.

Solution:
Let $P(n)$ be the statement $6 \mid 4^n + 7^n + 1$.
$P(1)$ is true: $P(1)$ states that $6 \mid 4^1 + 7^1 + 1$, or $6 \mid 12$, which is true.

$P(k) \to P(k+1)$: Suppose k is an integer for which $P(k)$ is true. That is, $6 \mid 4^k + 7^k + 1$. We need to show that $P(k+1)$ is true; that is, that $6 \mid 4^{k+1} + 7^{k+1} + 1$.

To show that $P(k+1)$ is true, we need to somehow connect $4^k + 7^k + 1$ and $4^{k+1} + 7^{k+1} + 1$. But

$$4^{k+1} + 7^{k+1} + 1 = 4\cdot4^k + 7\cdot7^k + 1$$
$$= (4^k + 3\cdot4^k) + (7^k + 6\cdot7^k) + 1$$
$$= 4^k + 7^k + 1 + 3\cdot4^k + 6\cdot7^k$$
$$= 4^k + 7^k + 1 + 3\cdot2^{2k} + 6\cdot7^k$$
$$= (4^k + 7^k + 1) + (3\cdot2\cdot2^{2k-1}) + (6\cdot7^k).$$

But each of the terms in parentheses is divisible by 6 — $P(k)$ guarantees that the first term in parentheses is divisible by 6, and the second and third terms in parentheses are each multiples of 6. Therefore, $P(k+1)$ is true.

Because $P(1)$ and $P(k) \to P(k+1)$ are true, by the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

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p.321, icon at Example 8
#2. Use the Principle of Mathematical Induction to prove that $2 \,|\, (n^2 - n)$ for all $n \geq 0$.

Solution:
Let $P(n)$ be $2 \,|\, (n^2 - n)$.

BASIS STEP: $P(0)$ is true because it is the statement $2 \,|\, 0^2 - 0$, or $2 \,|\, 0$.

INDUCTION STEP: $P(k) \to P(k+1)$: Suppose $P(k)$ is true; i.e., $2 \,|\, (k^2 - k)$.
Then $(k+1)^2 - (k+1) = k^2 + 2k + 1 - k - 1 = (k^2 - k) + 2k$. But $2 \,|\, (k^2 - k)$ by $P(k)$, and $2 \,|\, 2k$ because $2k$ is even. Therefore 2 is a divisor of the difference; i.e., $2 \,|\, (k+1)^2 - (k+1)$. Hence $P(k+1)$ is true.

We have shown that $P(0)$ is true and that $P(k) \to P(k+1)$ is true for all $k \geq 0$. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 0$.

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p.321, icon at Example 8
#3. Use mathematical induction to prove for all positive integers n, then $3^{2^n} - 1$ is divisible by 2^{n+2}.

Solution:
BASIS STEP: The basis step follows because $3^{2^1} - 1 = 3^2 - 1 = 8$ is divisible by $2^{1+2} = 2^3 = 8$.

INDUCTIVE STEP: The inductive hypothesis is the assumption that $3^{2^k} - 1$ is divisible by 2^{k+2}. Using this assumption, we need to show that $3^{2^{k+1}} - 1$ is divisible by 2^{k+3}. To show this, note that $3^{2^{k+1}} - 1 = 3^{2^k \cdot 2} - 1 = (3^{2^k})^2 - 1 = (3^{2^k} - 1)(3^{2^k} + 1)$. By the inductive hypothesis, the first term in this last product is divisible by 2^{k+2}. The second term in this product is even because 3^{2^k} is odd. We conclude (by the fundamental theorem of arithmetic) that this product is divisible by $2^{k+2} \cdot 2 = 2^{k+3}$. This completes the inductive step and the proof.

p.321, icon at Example 8

#4. Use mathematical induction to show that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct divisors of $n!$

Solution:

BASIS STEP: 1 can be expressed as the sum of the single term 1 and 1 is a divisor of $1! = 1$.

INDUCTIVE STEP: Assume that this statement holds for the positive integer k. That is, assume that every positive integer not exceeding $k!$ can be expressed as the sum of at most k distinct divisor of $k!$. Now consider a positive integer m not exceeding $(k+1)!$. By the division algorithm, we can write $m = q(k+1)+r$ where q and r are integers with $0 \le r \le k$. Moreover, because $0 < m < (k+1)!$ and $q = (m-r)/(k+1)$, we have $0 \le q \le k$. If $q = 0$, then $1 \le m = r \le k$ and by the inductive hypothesis, m is the sum of distinct divisors of $k!$, and consequently, the sum of distinct divisors of $(k+1)!$. If $q \ge 1$, then because $q \le k$, q can be written as sum of at most k distinct divisors of $k!$. That is,

$$q = d_1 + d_2 + \cdots + d_j$$

with $j \le k$ where $d_1, d_2, \ldots, d_j$ are distinct divisors of $k!$. It follows that

$$m = q(k+1) + r = d_1(k+1) + d_2(k+1) + \cdots + d_j(k+1) + r.$$

This sum includes at most $j+1$ terms. Because $r < k+1$, the terms in this sum are distinct and each term divides $(k+1)!$, unless $r = 0$ in which case we omit it in the sum.

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p. 322, icon at Example 9

1. Use mathematical induction to prove that $5^n + 9^n + 6$ is divisible by 4 whenever n is a positive integer.

Solution.

Let $P(n)$ be the statement that $5^n + 9^n + 6$ is divisible by 4.

BASIS STEP: To complete the basis step, we msut show that $P(1)$ is true, that is, that $5^1 + 9^1 + 6$ is divisible by 4. This is true because $5^1 + 9^1 + 6 = 5 + 9 + 6 = 20 = 5 \cdot 4$.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary positive integer k, that is, that $5^k + 9^k + 6$ is divisible by 4. To complete the inductive step, we must show that when we assume the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $5^{k+1} + 9^{k+1} + 6$ is divisible by 4, is also true.

We will show that the d, difference of $5^{k+1} + 9^{k+1} + 6$ and $5^k + 9^k + 6$, is divisible by 4. Once we have done this, we can use the inductive hypothesis to show that $5^{k+1} + 9^{k+1} + 6$ is divisible by 4. Note that

$$d = (5^{k+1} + 9^{k+1} + 6) - 5(5^k + 9^k + 6) = 9 \cdot 9^k + 6 - 5 \cdot 9^k - 30 = 4 \cdot 9^k - 24 = 4(9^k - 6).$$

Hence, $d = (5^{k+1} + 9^{k+1} + 6) - (5^k + 9^k + 6)$ is divisible by 4. Recall that the induction hypothesis tells us that $5^k + 9^k + 6$ is divisible by 4. Hence, $5^{k+1} + 9^{k+1} + 6 = d + (5^k + 9^k + 6)$ is also divisible by 4 because it is the sum of two terms each divisible by 4. This completes the inductive proof.

p.325, icon at Example 13

#1. Prove that for all positive integers n, $\dfrac{(2n)!}{2^n n!}$ is odd.

Solution:

For each positive integer n, let $P(n)$ be the statement $\dfrac{(2n)!}{2^n n!}$ is odd. To give a proof using the Principle of Mathematical Induction, we need to show that $P(1)$ is true and that $P(k) \rightarrow P(k+1)$ is true for all $k \geq 1$.

$P(1)$ is true: $P(1)$ states that $\dfrac{(2\cdot 1)!}{2^1 \cdot 1!}$ is odd. But $\dfrac{(2\cdot 1)!}{2^1 \cdot 1!} = \dfrac{2}{2} = 1$, which is odd.

$P(k) \rightarrow P(k+1)$ is true: Suppose that k is an integer such that $\dfrac{(2k)!}{2^k k!}$ is odd. We need to show that $\dfrac{(2(k+1))!}{2^{k+1}(k+1)!}$ is odd.

But $\dfrac{(2(k+1))!}{2^{k+1}(k+1)!} = \dfrac{(2k+2)!}{2^{k+1}(k+1)!} = \dfrac{(2k+2)(2k+1)(2k)!}{2\cdot 2^k \cdot (k+1)\cdot k!} = \dfrac{2(k+1)(2k+1)(2k)!}{2\cdot(k+1)\cdot 2^k \cdot k!} = (2k+1) \cdot \dfrac{(2k)!}{2^k \cdot k!}$.

The factor $2k+1$ is odd because of its form, and the second factor is odd by the assumption $P(k)$. We have a product of two odd integers. Therefore, the product is odd.

The statement $P(1)$ and $P(k) \rightarrow P(k+1)$ are both true. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all positive integers.

8

p.325, icon at Example 13

#2. *Muddy Children Puzzle*: The teacher of a group of children tells these children to play in their schoolyard without getting dirty. However, while playing, exactly n children get mud on their foreheads. When the children come back to the classroom after playing, the teacher states: "At least one of you has a muddy forehead" and then asks the children to answer "Yes" or "No" to the question: "Do you know whether you have a muddy forehead?"

The teacher asks this question over and over. What will the children with the muddy foreheads answer each time this question is asked, assuming that a child can see whether other children have muddy foreheads, but cannot see his or her own forehead? Furthermore, we assume that each child is honest, intelligent, and perceptive, and that all children answer each question simultaneously. (Note that the reason that the answer can differ when the question is asked repeatedly is that the children might learn from the previous answers of the other children.)

Solution:

We will use the Principle of Mathematical Induction to prove the following proposition:

$P(n)$: "the n children with muddy foreheads will answer 'No' each of the first $n-1$ times
   the question is asked, but all the children with muddy foreheads will answer 'Yes' the nth time."

BASIS STEP: Suppose that $n = 1$. The first time the question is asked, the sole child with a muddy forehead concludes that his or her forehead must be muddy because the teacher has announced that "At least one of you has a muddy forehead" and no other child has a muddy forehead. Consequently, this child answers "Yes."

Before turning to the inductive step, to gain some insight we consider the cases $k = 2$ and $k = 3$.

First, suppose that $k = 2$. Each of the two muddy children will answer "No" the first time the question is asked because of the mud on the other muddy child's forehead. Once the other muddy child has answered "No", each of the two muddy children knows that his or her forehead is dirty, because if it were clean the other child with a dirty forehead would have answered "Yes" the first time.

Next suppose that $k = 3$. One of the three children with mud on his or her forehead reasons as follows. Suppose that my forehead is clean. Then there are just two muddy children and by the $k = 2$ case both of these children will answer "Yes" the second time the question is asked. When they do not, this child knows that his or her forehead must be dirty. Consequently, all three children answer "Yes" the third time the question is asked.

INDUCTIVE STEP: Assume that $P(k)$ is true, that is, that if precisely k children have muddy foreheads, the first $k-1$ times the question is asked, the muddy children will answer "No", but the kth time it is asked, they all answer "Yes". Now assume that there are $k+1$ children with muddy foreheads.

A child with a muddy forehead will figure out how to answer the $(k+1)$st question as follows. Suppose that my forehead is clean, so that there are exactly k children with muddy foreheads, namely all the other k children whom I see have muddy foreheads. Then by the induction hypothesis, this child expects that the children with muddy foreheads will answer "No" the first $k-1$ times the question is asked, but "Yes" the kth time it is asked. But when this does not happen, the child with the muddy forehead concludes that his or her forehead is muddy.

Consequently, the $k+1$ children with muddy foreheads all answer "Yes" the $(k+1)$st time the question is asked.

We have shown that $P(1)$ is true and that $P(k) \to P(k+1)$ is true for all $n \geq 1$. Therefore, by the Principle of Mathematical Induction, the statement $P(n)$ is true for all $n \geq 1$.