

Rosen, Discrete Mathematics and Its Applications, 7th edition
Extra Examples
Section 6.1—The Basics of Counting



— Page references correspond to locations of Extra Examples icons in the textbook.

p.386, icon before Example 1

#1. There are three available flights from Indianapolis to St. Louis and, regardless of which of these flights is taken, there are five available flights from St. Louis to Dallas. In how many ways can a person fly from Indianapolis to St. Louis to Dallas?

Solution:

There are three ways to make the first part of the trip and five ways to continue on with the second part of the trip, regardless of which flight was taken for the first leg of the trip. Therefore, by the product rule there are $3 \cdot 5 = 15$ ways to make the entire trip.

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#2. A certain type of push-button door lock requires you to enter a code before the lock will open. The lock has five buttons, numbered 1, 2, 3, 4, 5.

- (a) If you must choose an entry code that consists of a sequence of four digits, with repeated numbers allowed, how many entry codes are possible?
- (b) If you must choose an entry code that consists of a sequence of four digits, with no repeated digits allowed, how many entry codes are possible?

Solution:

- (a) We need to fill in the four blanks in `__ __ __ __`, where each blank can be filled in with any of the five digits 1, 2, 3, 4, 5. By the generalized product rule this can be done in $5^4 = 625$ ways.
 - (b) We need to fill in the four blanks in `__ __ __ __`, but each blank must be filled in with a distinct integer from 1 to 5. By the generalized product rule that can be done in $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways.
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#3. Count the number of print statements in this algorithm:

```
for i := 1 to n
begin
  for j := 1 to n
    print "hello"
  for k := 1 to n
    print "hello"
end
```

Solution:

For each value of i , both the j -loop and k -loop are executed. Thus for each i , the number of print statements executed is $n + n$, or $2n$. Therefore the total number of print statements executed is $n \cdot 2n = 2n^2$.

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#4. Count the number of print statements in this algorithm:

```

for  $i := 1$  to  $n$ 
begin
  for  $j := 1$  to  $i$ 
    print "hello"
  for  $k := i + 1$  to  $n$ 
    print "hello"
end

```

Solution:

For each value of i , both the j -loop and k -loop are executed. Thus for each i , the number of print statements executed is i in the first loop plus $n - i$ in the second loop. Therefore, for each i , the number of print statements is $i + (n - i) = n$.

Therefore the total number of print statements executed is $n \cdot n = n^2$.

p.391, icon at Example 15

#1. Find the number of strings of length 10 of letters of the alphabet, with no repeated letters, that contain no vowels.

Solution:

The key to solving the problem is keep in mind a row of ten blanks: .

Each of the ten blanks in the string must contain one of the 21 consonants, with no repeated consonants allowed. By the extended version of the product rule, the answer is $21 \cdot 20 \cdot 19 \cdots 13 \cdot 12$.

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#2. Find the number of strings of length 10 of letters of the alphabet, with no repeated letters, that begin with a vowel.

Solution:

Keep in mind a row of ten blanks: .

There are five ways in which the first letter in the string can be a vowel. Once the vowel is placed in the first blank, there are 25 ways in which to fill in the second blank, 24 ways to fill in the third blank, etc. Using the extended product rule we obtain

$$\underbrace{5}_{\text{place vowel}} \cdot \underbrace{25 \cdot 24 \cdot 23 \cdots 18 \cdot 17}_{\text{place other letters}} .$$

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#3. Find the number of strings of length 10 of letters of the alphabet, with no repeated letters, that have C and V at the ends (in either order).

Solution:

Using a row of ten blanks, we first count the number of ways to have the pattern

$$\underline{C} \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{V}.$$

The number of ways to fill in the eight interior letters is $24 \cdot 23 \cdots 18 \cdot 17$.

Similarly, the number of words of the form

$$\underline{V} \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{C}$$

is $24 \cdot 23 \cdots 18 \cdot 17$.

Therefore, by the sum rule the answer is $(24 \cdot 23 \cdots 18 \cdot 17) + (24 \cdot 23 \cdots 18 \cdot 17) = 2(24 \cdot 23 \cdots 18 \cdot 17)$.

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#4. Find the number of strings of length 10 of letters of the alphabet, with repeated letters allowed, that have vowels in the first two positions.

Solution:

Keep in mind a row of ten blanks: $\text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ }.$

If vowels must be in the first two positions and letters can be repeated, we obtain the product

$$\underbrace{5 \cdot 5}_{\substack{\text{place 2} \\ \text{vowels}}} \cdot \underbrace{26 \cdot 26 \cdots 26}_{\substack{\text{place any} \\ \text{8 letters}}},$$

which is $5^2 \cdot 26^8$.

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#5. Find the number of strings of length 10 of letters of the alphabet, with no repeated letters, that have vowels in the first two positions.

Solution:

Keep in mind a row of ten blanks: $\text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ }.$

We will first count the number of ways to place vowels in the first two blanks. We can choose any of the five vowels for the first blank and any of the remaining four vowels to put in the second blank. Because we must do both, there are $5 \cdot 4 = 20$ ways to place the two vowels.

Next we will place eight of the remaining 24 letters in the remaining eight blanks. This can be done in $24 \cdot 23 \cdots 18 \cdot 17$ ways.

Therefore, by the product rule, the number of ways to place vowels in the first two blanks and eight letters in the remaining eight blanks is

$$\underbrace{(5 \cdot 4)}_{\substack{\text{place 2} \\ \text{vowels}}} \cdot \underbrace{(24 \cdot 23 \cdots 18 \cdot 17)}_{\substack{\text{place 8} \\ \text{other letters}}}.$$

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#6. Ten men and ten women are to be put in a row. Find the number of possible rows.

Solution:

Keep in mind a row of twenty blanks. There is no restriction on how the men and women can be placed in a row, so the answer is $20 \cdot 19 \cdot 18 \cdots 3 \cdot 2 \cdot 1 = 20!$.

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#7. Ten men and ten women are to be put in a row. Find the number of possible rows if no two of the same sex stand adjacent.

Solution:

Keep in mind a row of twenty blanks. If no two of the same sex stand in adjacent positions, then there are two possible patterns, using M for male and F for female:

MFMFMFMFMFMFMFMFMFMF and FMFMFMFMFMFMFMFMFMFM.

We will count the number of ways of achieving the first pattern and double this number to find the final answer. The first man can be chosen in 10 ways, the first woman can be chosen in 10 ways, the second man can be chosen in 9 ways, etc. Thus, by the extended product rule, each pattern can be obtained in

$$\underbrace{10}_{\substack{\text{place} \\ \text{first} \\ \text{man}}} \cdot \underbrace{10}_{\substack{\text{place} \\ \text{first} \\ \text{woman}}} \cdot \underbrace{9}_{\substack{\text{place} \\ \text{second} \\ \text{man}}} \cdot \underbrace{9}_{\substack{\text{place} \\ \text{second} \\ \text{woman}}} \cdots \underbrace{2}_{\substack{\text{place} \\ \text{ninth} \\ \text{man}}} \cdot \underbrace{2}_{\substack{\text{place} \\ \text{ninth} \\ \text{woman}}} \cdot \underbrace{1}_{\substack{\text{place} \\ \text{tenth} \\ \text{man}}} \cdot \underbrace{1}_{\substack{\text{place} \\ \text{tenth} \\ \text{woman}}}$$

or $(10 \cdot 9 \cdots 2 \cdot 1)^2$ ways. The second pattern can be obtained in the same number of ways, so we double the answer to get $2(10 \cdot 9 \cdots 2 \cdot 1)^2 = 2 \cdot 10!^2$.

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#8. Ten men and ten women are to be put in a row. Find the number of possible rows if Beryl, Carol, and Darryl want to stand next to each other in some order (such as Carol, Beryl, and Darryl, or Darryl, Beryl, and Carol).

Solution:

Keep in mind a row of twenty blanks.

We first consider the arrangements where Beryl, Carol, and Darryl stand next to each other in that order. We begin by putting the 17 other people in a row, which can be done in $17!$ ways. No matter how the 17

are placed in a row, Beryl, Carol, and Darryl can either be inserted (in that order) between two of the 17, or else placed at one of the two ends. This is pictured in the following diagram, where the 17 x's represent the 17 people placed in a row. There are 18 places in which Beryl, Carol, and Darryl can be placed together — 16 places between the x's and two places on the two ends — marked by blanks.

_ x _ x _ x _ x _ x _ x _ x _ x _ x _ x _ x _ x _ x _ x _ x _

Therefore, the number of ways to place the 18 people, with Beryl, Carol, and Darryl next to each other (in that order) is

$$\underbrace{17!}_{\text{place other 17}} \cdot \underbrace{18}_{\text{spots for B,C,D}} .$$

But Beryl, Carol, and Darryl can be permuted in $3!$ ways. Therefore, the final answer is $17! \cdot 18 \cdot 3!$.

p.393, icon at Example 18

#1. Find the number of integers from 1 to 400 inclusive that are:

- (a) divisible by 6.
- (b) not divisible by 6.

Solution:

(a) Every sixth integer from 1 to 400 is divisible by 6, yielding $\lfloor \frac{400}{6} \rfloor = 66$. (Note: When we divide 400 by 6 we obtain $66 \frac{2}{3}$. The fraction $\frac{2}{3}$ indicates that we are two thirds of the way toward the next integer divisible by 6, which is 402. We round down because 402 is beyond the range of numbers from 1 to 400.)

(b) From part (a) we know that 66 integers from 1 to 400 are divisible by 6. Hence the number that are not divisible by 6 is $400 - 66 = 334$.

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#2. Find the number of integers from 1 to 400 inclusive that are:

- (a) divisible by 6 and 8.
- (b) divisible by 6 or 8.

Solution:

(a) Divisibility by a and b is the same as divisibility by the least common multiple of a and b . Therefore, divisibility by 6 and 8 is the same as divisibility by 24. The answer is $\lfloor \frac{400}{24} \rfloor = 16$.

(b) We cannot take the number of integers divisible by 6 and add to it the number of integers divisible by 8, because this would count integers such as 24 or 48 twice (because they are divisible by both 6 and 8). We need to use the inclusion-exclusion principle to avoid the “double counting”.

Let $A_1 = \{x \mid 1 \leq x \leq 400, x \text{ is divisible by } 6\}$ and $A_2 = \{x \mid 1 \leq x \leq 400, x \text{ is divisible by } 8\}$. We want $|A_1 \cup A_2|$. By the inclusion-exclusion principle we have

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= \lfloor \frac{400}{6} \rfloor + \lfloor \frac{400}{8} \rfloor - \lfloor \frac{400}{24} \rfloor \end{aligned}$$

$$= 66 + 50 - 16$$

$$= 100.$$

Note: When the word “or” appears in a counting problem, it is a wise strategy to consider using the inclusion-exclusion principle.

p.393, icon at Example 18

#3. Find the number of strings of length 10 of letters of the alphabet, with repeated letters allowed,

- (a) that begin with C and end with V.
- (b) that begin with C or end with V.

Solution:

(a) Keep in mind a row of ten blanks: .

If the string has the form

$$\underline{\text{C}} \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{\text{V}},$$

there are 26 ways to fill in each of the eight intermediate blanks. Hence there are 26^8 strings of this form.

(b) We need to count the number of strings that have the form

$$\underline{\text{C}} \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{\text{V}} \quad \text{or} \quad \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{\text{C}} \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{\text{V}}$$

The number of strings of each form is 26^9 . However, we cannot simply add the two numbers to get our answer. If we do this, we are counting the number of strings of the form $\underline{\text{C}} \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \underline{\text{V}}$ twice because each such string fits both patterns.

We need to use the inclusion-exclusion principle to avoid the double-counting. Let A_1 be the set of all strings of length ten that begin with C and let A_2 be the set of all strings of length ten that end with V. We want $|A_1 \cup A_2|$. But by the inclusion-exclusion principle,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$= 26^9 + 26^9 - 26^8.$$

(The term 26^8 was obtained as the size of $A_1 \cap A_2$, which is the set of all strings of length ten that begin with C and end with V.)
