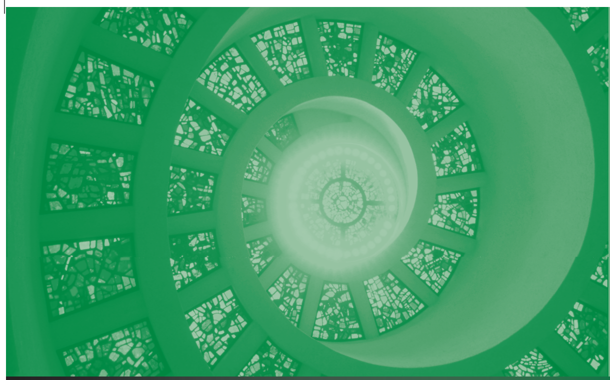

Chapter 10

Matrices

Topics covered

LEARNING OUTCOMES

By the end of this chapter you should understand:

1. The meaning of the term, 'matrix' and basic matrix notation
2. How to transpose a matrix
3. The meaning of row and column matrices and vectors
4. How to add and subtract matrices
5. Scalar matrices
6. How to multiply matrices
7. The determinant of a matrix and how to invert a matrix
8. Using inversions of 2×2 matrices to solve economic equations
9. How to calculate inversions for 3×3 matrices
10. How to use inversions of 3×3 matrices to solve economic equations
11. How to use Cramer's rule to solve economic equations.

10.1 Introduction

A matrix is another name for a table which can include data or other information. For example, the daily exchange rates which are published in most newspapers, railway timetables and election results all use simple tables or matrices to record useful information. We can use the idea of presenting information using rows and columns in mathematics, but the basic premise remains the same: we are trying to represent information in a simple format.

There are some straightforward rules which you will need to learn about how to use matrices. These rules will allow you to manipulate mathematical expressions and, more importantly, provide a further tool which can be used to analyse and solve economic problems and issues.

10.2 Definitions, notation and operations

A matrix comprises of n rows and m columns. A 3×4 matrix would have three rows and four columns. This describes the order of the matrix. In Table 10.1 the order is 2×3.

We frequently ignore the column and row headings and simply concentrate on the data in the table or matrix.

For example, consider an electronics multinational, Global Electrics PLC. The firm sells two products: an MP3 player (product A) and a car navigation system (product B). It sells these products in three countries: the UK, the USA and South Africa. The sales data is summarized in Table 10.1.

Table 10.1 Number of units sold of products A and B

	UK	USA	South Africa
MP3 player (product A)	2 million	12 million	1 million
Car navigation system (product B)	1 million	3 million	2 million

We can reduce this table to a much simpler matrix of information, which we will call 'G' to stand for Global Electrics PLC.

$$G = \begin{pmatrix} 2 & 12 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

The brackets indicate that we have a matrix of information. Each item within the matrix is called an entry or an element or a cell.

In a sense, a matrix can be seen as a sort of mathematical map: it sets out information over a set of rows and columns and, in the same way that we can use a grid reference to pinpoint a location on a geographic map, so we can use a special set of referencing or notation to identify a particular entry in the matrix.

For example, G_{12} means 'go to matrix G and record what it is in row 1 and column 2'. In our matrix G, the answer is 12 million. If you are familiar with Excel, this is very similar to the LOOKUP function and the process is identical.

Student note

A *matrix* is a table or array of information. The *matrix order* summarizes the number of rows and columns.

We use square or rounded brackets to show that we have a matrix of data and we identify particular entries using the following notation.

Generally, x_{ab} identifies the element in row *a*, column *b* in matrix X. The matrix below summarizes how the elements would be written in matrix X of the order 3 × 4:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix}$$

Quick problem 1

Consider the two matrices below, A and B.

$$A = \begin{pmatrix} 2 & 4 & 8 \\ 3 & 6 & 9 \\ 7 & 4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 9 & 6 & 0 \\ 3 & 11 & 8 & 1 \\ 5 & 2 & 0 & 10 \\ 7 & 4 & 2 & 9 \end{pmatrix}$$

Task One
Identify a_{22}, a_{33}, b_{11}, b_{42}.

Task Two
Write down the order of A and B.

Task Three
What problem would you face when trying to state b_{55}?

✔ **Learning outcome 1**

Transposing a matrix

Transposition really means ‘switching around’ and you might have come across this as a function within a spreadsheet program such as Excel. The switching means that all of the information in a row is placed in a column instead.

Student note

Transposition means replacing or switching around.

If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

and so on for whatever size or matrix order we have.

The superscripted T indicates that a transposition has taken place.

Consider:

$$A = \begin{pmatrix} 2 & 4 & 8 \\ 3 & 6 & 9 \\ 7 & 4 & 2 \end{pmatrix}$$

The transposition of matrix A involves taking the first column and rewriting it as a row and so on. We show that a transposition has taken place by placing a superscript 'T' next to the matrix so:

$$\text{Transposition of } A = A^T = \begin{pmatrix} 2 & 3 & 7 \\ 4 & 6 & 4 \\ 8 & 9 & 2 \end{pmatrix}$$

Quick problem 2

Task One

If $X = \begin{pmatrix} 12 & 11 & 3 & 7 \\ 3 & 2 & 1 & 1 \end{pmatrix}$

state X^T.

Task Two

Consider the following matrices, Y and Z:

$$Y = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 5 & 3 \\ 11 & 7 & 9 \end{pmatrix} \qquad Z = \begin{pmatrix} 2 & 2 & 4 \\ 3 & 4 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

Which statements, if any, are correct? Explain your reasoning in each case.

Statement 1: $Y^T = Z^T$

Statement 2: $Z^T = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 4 & 3 & 0 \end{pmatrix}$

Statement 3: $Y^T = \begin{pmatrix} 1 & 6 & 4 \\ 11 & 5 & 3 \\ 6 & 7 & 9 \end{pmatrix}$

✔ **Learning outcome 2**

10.3 Vectors

Vectors are a mathematical term which can be used to describe in a single line important information, e.g., speed and direction or an instruction to a computer on how to plot graphics. It has a wider use which we can apply in economics.

We can use a vector when information is contained in a single row or a single column.

For example, the matrix A = [2 1 3 0] would be a *single row vector* because all of the data is contained in a single row. Similarly, the matrix

$$B = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 7 \end{pmatrix}$$

would also be considered to be a vector but this time it would be labelled a *column vector* since all of the data is placed in one column.

Vectors are usually denoted by underlining the letter of the matrix and for historical reasons the lower case is always used. So, for example we would illustrate the row vector of matrix A simply by **a**. Equally, the column vector shown by matrix B would be denoted by **b**.

We can avoid the rather cumbersome expression of column vectors by simply rewriting them as a transposition. Look again at matrix B and consider how much easier and neater it is to simply write $\mathbf{b} = [1\ 2\ 2\ 7]^T$ – remember the T superscript tells us a transposition has taken place – rather than having to write out the whole column and waste a great deal of paper.

Quick problem 3

Task One
Write down how you can identify if a matrix is a vector.

Task Two
Which, if any, of the following are row vectors and column vectors?

$$H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \qquad J = (2 \quad 2 \quad 2 \quad -2) \qquad K = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^T \qquad L = (0)^T$$

✔ **Learning outcome 3**

10.4 Adding and subtracting matrices

So far we have used matrices to express a set of data. In economics, we often want to manipulate data to create new information. For example, we might want to add together household consumption in different regions to arrive at a figure for total national consumption. Equally, we might want to subtract elements of data, e.g., subtracting tax from gross income to work out a household's net income. Clearly, to complete these mathematical tasks we need to understand how to add and subtract matrices.

Student note

Matrices can be added or subtracted simply by adding or subtracting the matching entries.

In general, if we have two matrices A and B we can add them together to create a new matrix C as follows:

Adding matrices
If we have two matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 7 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 0 \\ 2 & 6 \end{pmatrix} \quad \text{then we can work out } A + B \text{ by:}$$

$$A + B = \begin{pmatrix} 2 & 3 \\ 7 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 2+3 & 3+0 \\ 7+2 & 4+6 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 9 & 10 \end{pmatrix}$$

So, in general:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix} \text{ then}$$

$$A + B = C = \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

Subtracting matrices

Subtracting matrices follows the same process: we subtract the second element from the first corresponding element. Using the same two matrices A and B as before:

$$A = \begin{pmatrix} 2 & 3 \\ 7 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 \\ 2 & 6 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2 & 3 \\ 7 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 2-3 & 3-0 \\ 7-2 & 4-6 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 5 & -2 \end{pmatrix}$$

So in general if:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix} \text{ then}$$

then:

$$A - B = \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

You will probably have already realized that you cannot add or subtract matrices which have a different order. For example, it would be impossible to add

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 0 \\ 3 & 3 & 2 \end{pmatrix}$$

because they do not share the same order: they do not have the same number of rows and columns.

Quick problem 4

Consider the following matrices

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 4 & 5 \\ 5 & 3 & 2 \\ 6 & 7 & 12 \end{pmatrix}$

$D = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $E = \begin{pmatrix} 8 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 9 & 3 \end{pmatrix}$ $F = (1 \quad 2)$ $G = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Evaluate:

(a) $A + B$

(b) $B + C$

(c) $G - E$

(d) $C + E$

(e) $E - C$

(f) $G - D$

(g) $A + E$

✔ **Learning outcome 4**

10.5 Dealing with 'zero matrices'

Sometimes an addition or subtraction will lead to a zero answer.

For example:

$$A - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E - E = \begin{pmatrix} 8 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 9 & 3 \end{pmatrix} - \begin{pmatrix} 8 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D - D = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In each case, the answer is a zero matrix and we can simplify and write **0**.

10.6 Scalar multiplication of matrices

Sometimes we want to multiply quantities by a factor. For example, a worker who earns £50 per day will need to multiply their daily wage rate by the number of days they work in a month to calculate their monthly earnings.

We can use this idea of scaling with matrices. In this context, 'scaling' means multiplying each element by a given number. The scaling number or 'scalar' can be positive or negative and it can be a fraction.

Example

If $A = \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}$, calculate

(a) $3A$ (b) $\frac{1}{2}A$ (c) $-3A$

(a) $3A = \begin{pmatrix} 3\times 2 & 3\times 3 \\ 3\times 4 & 3\times 2 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 12 & 6 \end{pmatrix}$

(b) $\frac{1}{2}A = \begin{pmatrix} \frac{1}{2}\times 2 & \frac{1}{2}\times 3 \\ \frac{1}{2}\times 4 & \frac{1}{2}\times 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 2 & 1 \end{pmatrix}$

(c) $-3A = \begin{pmatrix} -3\times 2 & -3\times 3 \\ -3\times 4 & -3\times 2 \end{pmatrix} = \begin{pmatrix} -6 & -9 \\ -12 & -6 \end{pmatrix}$

Student note

Scaling a matrix means multiplying each element by a certain factor but remember that this can be a positive, negative, fractional or whole number.

If we have a matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and we want to find kA where k is the scalar then we multiple each element by k:

$$kA = \begin{pmatrix} k\times a_{11} & k\times a_{12} \\ k\times a_{21} & k\times a_{22} \end{pmatrix}$$

If:

$k > 1$ then each element in the matrix kA will be bigger than each element in A

$k = 1$ then each element in the matrix kA will identical to each element in A

$k < 1$ then each element in the matrix kA will smaller than each element in A

$k < 0$ then each element in the matrix kA will be negative

Quick problem 5

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad Z = \begin{pmatrix} 3 & 7 & 6 \\ 4 & 7 & 3 \\ 5 & -2 & 7 \end{pmatrix}$$

Task One
Evaluate:

(a) 2X

(b) 3Y

(c) $\frac{1}{2}Z$

Task Two
Which statement, if any, is true?

Statement 1: 4X = 2(2X)

Statement 2: X − Y = 0

Statement 3: $3Z = \begin{pmatrix} 9 & 21 & 18 \\ 12 & 14 & 6 \\ 15 & 6 & 21 \end{pmatrix}$

Statement 4: $5Y = \begin{pmatrix} 12 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 16 \end{pmatrix}$

✔ **Learning outcome 5**

10.7 Multiplying matrices

In the previous section we considered how to scale a matrix by a given factor. The next step is to think about how we can multiply matrices together. Like many mathematical processes you will need to learn a certain technique for doing this but first we can think about why we might want to do this at all.

There are many practical examples including calculating a firm's revenue or turnover, the total dividend payment a shareholder should expect to receive or measuring the total output of an economy by multiplying the average output per workers in sectors of the economy by the total number of workers in each sector.

We will look at a firm's profits to illustrate matrix multiplication. Imagine a large mobile phone manufacturer examines the profits on three models of mobile phone that it produces: the A100, the A101 and the A102. The features, profits per phone and quantity sold are summarized in Table 10.2.

Table 10.2

	A100	A101	A102
Profit per mobile phone sold	£15.00	£18.00	£23.00
Main features:			
MP3 player	✘	✔	✔
Video messenging	✘	✘	✔
FM radio	✘	✔	✔
Bluetooth	✔	✔	✔

We are also give the following data on the worldwide quantity sold last year of each model:

A100: 100 million units sold

A101: 75 million units sold

A102: 130 million units sold

We can express the numerical data in simple matrices.

Let P denote the profit on each model of mobile phone A100, A101 and A102 respectively. That is,

$$P = \begin{pmatrix} 15 & 18 & 23 \end{pmatrix}$$

Let Q denote the quantity of each model sold. That is,

$$Q = \begin{pmatrix} 100 \\ 75 \\ 130 \end{pmatrix}$$

If we want to work out total profits for each model, we need to multiply P by Q:

$$P \times Q = \begin{pmatrix} 15 & 18 & 23 \end{pmatrix} \times \begin{pmatrix} 100 \\ 75 \\ 130 \end{pmatrix}$$

We need a quick technique to do this. In effect, we need to multiply the 15 by the 100, then multiply the 18 by the 75 and finally multiply the 23 by the 130. The scalar product we would arrive at would show the total profit attained in selling the three mobile phone models. That is, we need to calculate as follows:

$$P \times Q = \begin{pmatrix} 15 & 18 & 23 \end{pmatrix} \times \begin{pmatrix} 100 \\ 75 \\ 130 \end{pmatrix}$$

Or, put another way

$$P \times Q = (15 \times 100 + 18 \times 75 + 23 \times 130) = (1500 + 1350 + 2990) = £5840$$

Student note

Matrices can be multiplied together if the number of rows in one matrix is the same as the number of columns in the other matrix.

We can multiply the two matrices together by taking the first row of the first matrix and placing it over the top of the first column of the second matrix. We then multiply the each pair of elements and add together.

For example if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and want to calculate AB we take the first row of A and multiply each element against corresponding element of the first column of matrix B and add together.

So AB is then:

$$AB = \begin{pmatrix} (a_{11} \times b_{11}) + (a_{12} \times b_{21}) & (a_{11} \times b_{12}) + (a_{12} \times b_{22}) \\ (a_{21} \times b_{11}) + (a_{22} \times b_{21}) & (a_{21} \times b_{12}) + (a_{22} \times b_{22}) \end{pmatrix}$$

Example 1

$$\text{Let: } A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} (2 \times 2) + (3 \times 1) & (2 \times 1) + (3 \times (-1)) \\ (4 \times 2) + (5 \times 1) & (4 \times 1) + (5 \times (-1)) \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 13 & -1 \end{pmatrix}$$

Example 2

$$\text{Let } C = (2 \quad 6 \quad 3) \quad \text{and} \quad D = \begin{pmatrix} 1 & 4 & 2 \\ 5 & 0 & 6 \\ -1 & 7 & 9 \end{pmatrix}$$

$$CD = ((2 \times 1) + (6 \times 5) + (3 \times (-1)) \quad (2 \times 4) + (6 \times 0) + (3 \times 7) \quad (2 \times 2) + (6 \times 6) + (3 \times 9))$$

$$CD = (2 + 30 - 3 \quad 8 + 0 + 21 \quad 4 + 36 + 27)$$

$$CD = (29 \quad 29 \quad 67)$$

Example 3

$$X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 & 2 \\ 3 & 7 \\ 0 & 3 \\ 5 & -2 \end{pmatrix}$$

$$XY = \begin{pmatrix} (1 \times 1) + (2 \times 3) + (3 \times 0) + (4 \times 5) & (1 \times 2) + (2 \times 7) + (3 \times 3) + (4 \times (-2)) \\ (5 \times 1) + (6 \times 3) + (7 \times 0) + (8 \times 5) & (5 \times 2) + (6 \times 7) + (7 \times 3) + (8 \times (-2)) \end{pmatrix}$$

$$XY = \begin{pmatrix} 27 & 17 \\ 63 & 57 \end{pmatrix}$$

Quick problem 6

You are given the following economic data.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & 5 \\ -1 & 3 & 0 \\ 6 & -2 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 4 & 1 & -2 \end{pmatrix}$$

Find the following. Identify which, if any, cannot be found and why this is the case.

(a) AD

(b) BE

(c) AF

(d) DC

✔ **Learning outcome 6**

10.8 Matrix inversion

The word 'inversion' simply means to turn something upside down. If, for example, we invert the number '4' we get '$^1/_4$': we have turned the number upside down or calculated its reciprocal and we did this by calculating 4^{-1} to arrive at $^1/_4$. In terms of matrices we can do something similar but we need to learn a special technique for doing this.

Consider the following, matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we want to find the inverse of A or A^{-1}.

We need to follow three separate steps:

Step 1: Swap elements *a* and *d*.

Step 2: Change the signs of elements *b* and *c*.

Step 3: Multiple the matrix we now have after steps 1 and 2 by

$$\frac{1}{(ad - bc)}.$$

The resulting answer gives us A^{-1}.

Example

If $A = \begin{pmatrix} 1 & -4 \\ 0 & 3 \end{pmatrix}$ then calculate A^{-1}.

Using our three-step method:

Step 1 gives us:

$$\begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}$$

Step 2 gives us:

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$$

Step 3 gives us:

$$\frac{1}{3} \times \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4/3 \\ 0 & 1/3 \end{pmatrix}$$

So, if

$$A = \begin{pmatrix} 1 & -4 \\ 0 & 3 \end{pmatrix}$$

then

$$A^{-1} = \begin{pmatrix} 1 & 4/3 \\ 0 & 1/3 \end{pmatrix}$$

Note just as $4 \times {}^1/_4 = 1$ that $A \times A^{-1}$ equals the identity matrix.

$$A \times A^{-1} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 4/3 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Student note

In step 3, we used a special scalar

$$\frac{1}{(ad - bc)}$$

The term $(ad - bc)$ is known as *the determinant of the matrix*. In shorthand, we often write this as *det(A)* to refer to the determinant of matrix A.

You can easily see that if $(ad - bc)$ or det(A) is zero we cannot work the inversion because the scalar

$$\frac{1}{(ad - bc)}$$

would be infinity (remember: any number divided by zero cannot be worked out).

What does this mean in practice? It means that if:

$(ad - bc) = 0$ then the matrix is *a singular matrix, i.e., it cannot be inverted.*

$(ad - bc) \neq 0$ then the matrix is *a non-singular matrix, i.e., it can be inverted.*

It is worth checking whether $\det(A) = 0$ before you start trying to invert it. If det(A) is zero you know not to bother trying to invert the matrix: it cannot be done!

The identity matrix

You will have seen from the worked example a special matrix called the *identity matrix* which looks like this:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Quick problem 7

If:

$$A = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & -1 \\ -2 & 7 \end{pmatrix}$$

Task One

(a) Calculate det(A), det(B) and det(C)

(b) Which matrices are singular or non-singular?

Task Two

Using your answers from Task One, calculate A^{-1}, B^{-1}, C^{-1}.

✔ **Learning outcome 7**

10.9 Application of 2 × 2 matrices and inversions: solving economic equations

We can use this technique for solving economics problems. A common use is to solve problems involving prices of goods. For example, if we are told that:

$$aP_1 + bP_2 = e$$

$$cP_1 + dP_2 = f$$

how can we solve this to find the values of P_1 and P_2?

It is worth noting a general rule first before we start to solve problems. In this format we can see that we can form a matrix, A, with the values of a, b, c and d. We could multiply this by the P values P_1 and P_2 to get the values on the right-hand side of each equation e and f.

We can put this another way.

Let:

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad x = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} e \\ f \end{pmatrix}$$

Then:

$$\mathrm{A} \times x = z$$

$$\mathrm{A}x = z$$

You need to learn this can be rearranged to form:

$$x = \mathrm{A}^{-1}z$$

where A^{-1} is the inverse of matrix A.

Example

This process can be useful when solving economic problems with equations. Consider the economic data below concerning two commodities, tea and coffee.

Let:

P_1 denote the equilibrium price of coffee

P_2 denote the equilibrium price of tea

An economist finds that:

$$4P_1 - P_2 = 1$$

$$-2P_1 + 3P_2 = 12$$

and he wants to find the values of P_1 and P_2.

We need to rewrite these two equations into matrix form. Using the structure above we can see that:

$$A = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$$

$$x = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$z = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

We know that $x = A^{-1}z$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

$$x = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

Therefore,

$$x = \begin{pmatrix} 3/10 & 1/10 \\ 1/5 & 3/5 \end{pmatrix}\begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

$$= \begin{pmatrix} 3/10 & 12/10 \\ 1/5 & 24/5 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 5 \end{pmatrix}$$

Therefore, $P_1 = 1.5$ and $P_2 = 5$

The economist finds that the equilibrium price of coffee is 1.5 and the equilibrium price of tea is 5.

Student note

When using matrices to solve simple equations remember to reformulate the equations so they follow this structure:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad x = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \text{ and } z = \begin{pmatrix} e \\ f \end{pmatrix}$$

Then:

$$x = A^{-1}z$$

To get A^{-1} you'll need to follow the three-step process outlined at the beginning of section 10.8.

Quick problem 8

Task One

An economist believes that the equilibrium prices of two goods p_a and p_b are linked as follows.

$p_a - 3p_b = 3$

$5p_a - 9p_b = 11$

Assuming this data is correct, find the values of p_a and p_b using the matrix method.

Task Two

Look again at tour answers to Task One. What evidence is there that the economist might be using incorrect data?

Task Three

The economist finds new information which suggests the relationship between p_a and p_b has altered over time and can now be expressed as follows

$p_a + 3p_b = 1$

$2p_a - 4p_b = 1$

Do the values of p_a and p_b change? If so, what are they now?

✔ **Learning outcome 8**

10.10 Inversions of 3 × 3 matrices

We can expand this knowledge and understanding to larger matrices. Consider, for example, matrix A which has three rows and three columns.

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

To work out the inverse of this matrix we need to understand a new term: *cofactors*. Cofactors are a simple but powerful idea. If we remove the row and column which contains the element a_1 we are left with a 2 × 2 matrix. The *determinant* of this sub-matrix gives us the *minor of* a_1.

So, the minor of $a_1 = (b_2c_3 - c_2b_3)$.

Equally, the minor of $c_1 = (a_2b_3 - b_2a_3)$

The cofactor of each element is calculated by multiplying the minor by ± 1 using the following structure or pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

We can create a new matrix by replacing each element of matrix A by its cofactor and *transposing* this to find the *adjoint of* A. This is sometimes written as *adj*(A).

We need to then calculate the determinant of A. The determinant of A in a 3 × 3 matrix can be worked out by multiplying each member or element in a row or column by their respective cofactors and adding them together.

We can then calculate

$$A^{-1} = \text{adj}(A) \times \frac{1}{\det(A)}$$

Student note

In summary, to work out the inverse of a 3 × 3 matrix there are five steps:

Step 1: Work out the minors of each element in our matrix

Step 2: Calculate the matrix of cofactors

Step 3: Create the adjoint of A

Step 4: Calculate the det(A)

Step 5: Work out

$$\text{adj}(A) \times \frac{1}{\det(A)} \text{ arrive at } A^{-1}$$

Full example

We are given matrix A as follows and asked to calculate A^{-1}.

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Step 1: Work out the minors

The minors of each element in matrix A are as follows:

$$\text{Minor of } a_1 = \text{determinant of the matrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (1 \times 1) - (0 \times 1) = 1$$

$$\text{Minor of } a_2 = \text{determinant of the matrix} \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} = (-2 \times 1) - (3 \times (-1)) = 1$$

$$\text{Minor of } a_3 = \text{determinant of the matrix} \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} = -3$$

$$\text{Minor of } b_1 = \text{determinant of the matrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 2$$

Minor of b_2 = determinant of the matrix $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = -2$

Minor of b_3 = determinant of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = -6$

Minor of c_1 = determinant of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -3$

Minor of c_2 = determinant of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} = 1$

Minor of c_3 = determinant of the matrix $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = 5$

The matrix of minors is then:

$$\begin{pmatrix} 1 & 1 & -3 \\ 1 & -2 & 1 \\ -3 & -6 & 5 \end{pmatrix}$$

Step 2: Calculate the cofactors
We multiply the matrix of minors by

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Matrix of cofactors is then:

$$\begin{pmatrix} 1 & -2 & -3 \\ -1 & -2 & -1 \\ -3 & 6 & 5 \end{pmatrix}$$

Step 3: Create the adjoint of A by transposing the matrix of cofactors
From Step 2 we transpose this matrix to get:

$$\text{adj}(A) = \begin{pmatrix} 1 & -1 & -3 \\ -2 & -2 & 6 \\ -3 & -1 & 5 \end{pmatrix}$$

Step 4: Calculating the det(A)
The determinant of A is calculated by multiplying each member or element in any row or column by their respective cofactors and adding them together.

In other words take each element in Matrix A

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

and multiply by the corresponding element in the matrix cofactors

$$\begin{pmatrix} 1 & -2 & -3 \\ -1 & -2 & -1 \\ -3 & 6 & 5 \end{pmatrix}$$

Looking at the first row we get: $(1 \times 1) + (-2 \times (-2)) + (3 \times (-3)) = 1 + 4 - 9 = -4$.

We get the same answer no matter which row or column we select. To prove this lets repeat this calculation of the determinant of A using the second row.

Looking at the second row we get: $(2 \times (-1)) + (1 \times (-2)) + (0 \times (-1)) = -4$.

Step 5: Multiplying adj(A) by $\frac{1}{\det(A)}$

Multiplying adj(A) by 1/det(A) we have:

$$A^{-1} = \text{adj}(A) \times \frac{1}{\det(A)}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & -3 \\ -2 & -2 & 6 \\ -3 & -1 & 5 \end{pmatrix} \times \left(-\frac{1}{4}\right)$$

or,

$$A^{-1} = -\frac{1}{4}\begin{pmatrix} 1 & -1 & -3 \\ -2 & -2 & 6 \\ -3 & -1 & 5 \end{pmatrix}$$

Brief example

If

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

find A^{-1}.

Step 1

Minors are

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix}$$

Step 2

Cofactors are

$$\begin{pmatrix} -2 & -1 & 3 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

Step 3

$$\text{adj}(A) = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

Step 4

$$\det(A) = 1$$

Step 5

$$A^{-1} = \text{adj}(A) \times \frac{1}{\det(A)} = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -1 & 1 \\ 3 & 2 & -1 \end{pmatrix} \times 1 = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

Quick problem 9

Task One

Calculate the inverse of the following 3×3 matrices.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & -2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 5 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

Task Two

What problem is there in trying to find D^{-1} where

$$D = \begin{pmatrix} 4 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

✔ **Learning outcome 9**

10.11 Application of 3 × 3 matrices and inversions: solving economic equations

In the same way as in section 10.9, we can apply our knowledge and understanding of matrix algebra to solve economic equations. We can use the same statement that $x = A^{-1}z$ where A^{-1} is the inverse of matrix A.

Student note

We have already seen how to use matrices to solve for an economic problem where two variables are linked. To do this we needed to invert a 2 × 2 matrix. We then looked at how to invert a 3 × 3 matrix. The next step is to use this leaning to solve for an economic problem where three variables are linked.

In the example below we follow this process of inverting a 3 × 3 matrix to work out the values of three economic variables. We do this by working through the 'five-step process' outlined in the previous section.

Consider the example below which follows closely the process we used in 10.9 to solve for two sets of pricing data. This time we have three economic variables: P_1, P_2 and P_3 representing the *profits* which a manufacturer earns from the production and sale of tea, coffee and cocoa respectively.

Example

Let:

p_1 denote the profits from the production and sale of tea

p_2 denote the profits from the production and sale of coffee

p_3 denote the profits from the production and sale of cocoa

We find that:

$$2p_1 - p_2 + p_3 = 5$$

$$p_1 - 3p_2 + 2p_3 = 2$$

$$2p_1 + p_2 + 4p_3 = -3$$

We create the matrix A:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 4 \end{pmatrix}$$

$$x = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$z = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

We know that

$A \times x = z$

$Ax = z$

and that $x = A^{-1}z$ where A^{-1} is the inverse of matrix A.

Step 1
We calculate the minors to be:

$$\begin{pmatrix} -14 & 0 & 7 \\ -5 & 6 & 4 \\ 1 & 3 & -5 \end{pmatrix}$$

Step 2
We calculate the cofactors to be:

$$\begin{pmatrix} -14 & 0 & 1 \\ 5 & 6 & -4 \\ 1 & -3 & -5 \end{pmatrix}$$

Step 3
We work out adj(A) as:

$$\begin{pmatrix} -14 & 5 & 1 \\ 0 & 6 & -3 \\ 7 & -4 & -5 \end{pmatrix}$$

Step 4
We can see that det(A) can be calculated by taking the element of any row or column of A and multiplying each value by the cofactor obtained in step 2. Note using row 1:

$$2 \times (-14) + (-1) \times 0 + 1 \times 7 = -28 + 0 + 7 = -21$$

or using column 3:

$$1 \times 7 + 2 \times (-4) + 4 \times (-5) = 7 - 8 - 20 = -21$$

So the determinant of A is −21.

Step 5

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} = -\frac{1}{21} \times \begin{pmatrix} -14 & 5 & 1 \\ 0 & 6 & -3 \\ 7 & -4 & -5 \end{pmatrix}$$

$x = A^{-1}z$

$$\Rightarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\frac{1}{21} \times \begin{pmatrix} -14 & 5 & 1 \\ 0 & 6 & -3 \\ 7 & -4 & -5 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 2/3 & -5/21 & -1/21 \\ 0 & -2/7 & 1/7 \\ -1/3 & 4/21 & 5/21 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

$p_1 = 2^5/_7$

$p_2 = -^1/_7$

$p_3 = -^4/_7$

Given the economic data we have been given, the manufacturer will make a profit of $2^5/_7$ on tea but make a loss of $^1/_7$ on coffee and a loss of $^4/_7$ on cocoa.

We can verify that this is correct by substituting the values in the initial equations, as follows:

$2p_1 - p_2 + p_3 = 5$

$p_1 - 3p_2 + 2p_3 = 2$

$2p_1 + p_2 + 4p_3 = -3$

Quick problem 10

A multinational corporation produces a wide range of electronic products.

p_1 represents the profits from sales of a new DVD player

p_2 represents profits from the sales of a new plasma television

p_3 represents profits from the sales of an existing but popular hi-fi system.

Task One

The economics department believes that profits are linked as follows:

$p_1 + p_2 = 2$

$2p_2 - p_3 = -7$

$3p_1 + p_3 = 16$

Using a 3 × 3 matrix work out the profits of each product (p_1, p_2, and p_3).

Task Two

The economics department receives new data and believes that the relationship is now of the form:

$p_1 - 2p_2 + 3p_3 = -1$

$3p_1 + p_2 + p_3 = 2$

$2p_1 + p_2 - 3p_3 = -5$

Calculate the new values of p_1, p_2 and p_3 using a 3 × 3 matrix.

✔ **Learning outcome 10**

10.12 Cramer's rule

In previous sections we have looked at what matrices are and how they can be used to solve economic problems. We started by looking at 2×2 matrices and then by applying the idea of matrix inversion we were able to solve simple two variable problems. The next step involved extending this to a 3×3 matrix using the 'five-step technique'. We have built up a comprehensive set of tools for dealing with economic relationships involving two or three different variables.

We are now presented with a more general issue: can we find out a fundamental or general way of solving *any* matrix? At the moment we have confined our focus to only 2×2 and 3×3 matrices, but what if we have four or five economic variables? How could we solve these?

The practical need or significance of this might not be obvious. Economies are, in effect, very complex systems of variables such as consumption, investment, government spending, imports exports, taxation, interest rates and so on. Moreover, economic variables are commonly linked or related in some way: a household's consumption is usually linked to income, income is linked to factors such as wages, and the demand or consumption for goods and services produced in the first place. This sense of interconnectedness is captured in the simple circular flow of income model which is set out in most elementary economics textbooks.

If we think of an economy as a series of connections and relationships, it is perhaps easier to appreciate that economies can be expressed or articulated using mathematical expressions or functions. If economic variables are interconnected and linked, then we can also see that one way of thinking about an economy is a massive set of mathematical expressions. We can therefore use matrices to try and better understand what the values of economic variables are.

To move from 3×3 matrices to bigger systems we need a general rule or technique. One rule we will use is *Cramer's rule*, named after a Swiss mathematician who looked at large systems of mathematical relationships or equations.

Student note

Cramer's rule helps us to solve *any matrix with an equal number of rows and columns.*

First, we must a have a matrix with m rows and m columns.

Second, if we have a matrix A, a set of unknowns x and a set of knowns (see section 10.9) we need to recall that $A \times x = z$ and then $Ax = z$.

Third, Cramer's rule tells us that if we want to find a certain 'x variable' – let us call it x_i – then we can find it as follows:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

So what is meant by A_i?

A_i can be found by switching or substituting the ith column of our original matrix A with the right-hand side of our vector z.

In general terms and for 2×2 matrix if we have:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad x = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} e \\ f \end{pmatrix}$$

$$A_1 = \begin{pmatrix} e & b \\ f & d \end{pmatrix}$$

For A_1 we replace the *first* column in matrix A with the vector values from z: we swap '*a*' and '*c*' for '*e*' and '*f*'

and $A_2 = \begin{pmatrix} a & e \\ c & f \end{pmatrix}$

For A_2 we replace the *second* column in matrix A with the vector values from z: we swap '*b*' and '*d*' this time for '*e*' and '*f*'

We can use Cramer's rule to find out quickly the values of economic variables in a large matrix. It is useful because it is a relatively quick and neat way to get to the answers we want.

Example for a 2 × 2 matrix

We have price data on two commodities as follows:

$$3P_1 - 2P_2 = -3$$

$$4P_1 - P_2 = 7$$

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}, x = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, z = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

Using Cramer's rule:

$$P_1 = \frac{\det(A_1)}{\det(A)}$$

$$A_1 = \begin{pmatrix} -3 & -2 \\ 7 & -1 \end{pmatrix}$$ (we switch the '3' and the '4' for the '−3' and the '7')

$$\det(A_1) = (-3 \times (1)) - (-2 \times 7) = 3 + 14 = 17$$

$$\det(A) = (3 \times (-1)) - (-2 \times 4) = -3 + 8 = 5$$

$$\Rightarrow P_1 = \frac{17}{5} = 3.4$$

Using Cramer's rule:

$$P_2 = \frac{\det(A_2)}{\det(A)}$$

$$A_2 = \begin{pmatrix} 3 & -3 \\ 4 & 7 \end{pmatrix}$$ (we switch the second column this time)

$$\det(A_2) = (3 \times 7) - (-3 \times 4) = 21 + 12 = 33$$

We know from above that $\det(A) = 5$

$$P_2 = \frac{\det(A_2)}{\det(A)} = \frac{33}{5} = 6.6$$

Example for a 3 × 3 matrix

The principle is exactly the same for a 3 × 3 matrix although you might want to refer back to section 10.10 to remember how to calculate det(A) for a 3 × 3 matrix using cofactors etc.

We are told that:

$$p_1 + 2p_2 + 3p_3 = 17$$
$$3p_1 + 2p_2 + p_3 = 11$$
$$p_1 - 5p_2 + p_3 = -5$$

We set up using our usual notation:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{pmatrix}, x = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, z = \begin{pmatrix} 17 \\ 11 \\ -5 \end{pmatrix}$$

$$\begin{aligned}\det(A) &= 1 \times [(2 \times 1) - (1 \times (-5))] - 2 \times [(3 \times 1) - (1 \times 1)] + 3 \times [(3 \times (-5)) - (2 \times (-1))] \\ &= 7 - 4 - 51 = 48\end{aligned}$$

We will need to refer back to our answer for det(A).

Calculating p_1

Now,

$$A_1 = \begin{pmatrix} 17 & 2 & 3 \\ 11 & 2 & 1 \\ -5 & -5 & 1 \end{pmatrix} \quad \text{(replacing first column of A with vector z)}$$

$$\begin{aligned}\det(A_1) &= 17 \times [(2 \times 1) - (1 \times (-5))] - 2 \times [(11 \times 1) - (1 \times (-5))] + 3 \times [(11 \times (-5)) - (2 \times (-5))] \\ &= (17 \times 7) - (2 \times 16) + (3 \times (-45)) \\ &= 119 - 32 - 135 = -48\end{aligned}$$

Going back to Cramer's rule that

$$x_i = \frac{\det(A_i)}{\det(A)}$$

$$\Rightarrow p_1 = \frac{\det(A_1)}{\det(A)} = \frac{-48}{-48} = 1$$

Calculating p_2

$$A_2 = \begin{pmatrix} 1 & 17 & 3 \\ 3 & 11 & 1 \\ 1 & -5 & 1 \end{pmatrix}$$

$$\begin{aligned}\det(A_2) &= 1 \times [(11 \times 1) - (1 \times (-5))] - 17 \times [(3 \times 1) - (1 \times 1)] + 3 \times [(3 \times (-5)) - (11 \times 1)] \\ &= (1 \times 16) - (17 \times 2) + (3 \times (-26)) \\ &= 16 - 34 - 78 = -96\end{aligned}$$

$$p_2 = \frac{\det(P_2)}{\det(P)} = \frac{-96}{-48} = 2$$

Calculating p_3

$$A_3 = \begin{pmatrix} 1 & 2 & 17 \\ 3 & 2 & 11 \\ 1 & -5 & 5 \end{pmatrix}$$

$$\begin{aligned} \det(A_3) &= 1 \times [(2 \times (-5)) - (11 \times (-5))] - 2 \times [(3 \times (-5)) - (11 \times 1)] + 17 \times [(3 \times (-5)) - (2 \times 1)] \\ &= (1 \times 45) - (2 \times (-26)) + (17 \times (-17)) \\ &= 45 + 52 - 289 = -192 \end{aligned}$$

$$p_3 = \frac{\det(A_3)}{\det(A)} = \frac{-192}{-48} = 4$$

Quick problem 11

Task One

A statistician knows that C + I = Y where C = consumption, I = investment and Y = national income.

She reviews some recent economic data and concludes that

$7C + 3I = 15$

$-2C + 5I = 16$

(a) Using Cramer's rule find the values of C and I.

(b) Comment on your finding for I.

Task Two

The same statistician amends her work in light of revised data. She develops her economic model as follows:

$2C + 3I + G = 10$

$C - I + G = 4$

$4C - I - 5G = -8$

This model introduces government spending (G).

What are the values of C, I and G in this model of the economy?

Task Three

In another economy different economic equations are found as follows:

$3C - 2I + 2G = 4$

$-C + 3I + 2G = 2$

$2C + 4I - 6G = 1$

Use matrices and Cramer's rule to solve.

✔ **Learning outcome 11**

Answers to quick problems

Quick problem 1

Task One

$a_{22} = 6$

$a_{33} = 2$

$b_{11} = 1$

$b_{42} = 4$

Task Two

A is a 3×3 matrix.

B is a 4×4 matrix.

Task Three

b_{55} refers to the element in the fifth column and the fifth row. The matrix does not extend to a fifth column and/or a fifth row.

Quick problem 2

Task One

$$X^T = \begin{bmatrix} 12 & 3 \\ 11 & 2 \\ 3 & 1 \\ 7 & 1 \end{bmatrix}$$

Task Two

Statement 1: false

$$Y^T = \begin{bmatrix} 1 & 6 & 11 \\ 2 & 5 & 7 \\ 4 & 3 & 9 \end{bmatrix} \neq Z^T = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$

Statement 2: true

$$Z^T = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$

Statement 3: false

$$Y^T = \begin{bmatrix} 1 & 6 & 11 \\ 2 & 5 & 7 \\ 4 & 3 & 9 \end{bmatrix}$$

Quick problem 3

Task One

A vector can be identified if it is comprised of a single row *or* a single column of entries or elements.

Task Two

H, J and L are vectors: they have single rows/columns. Matrix L is a unique case: it has one entry and can be considered to be a single row or a single column.

Quick problem 4

(a) $A + B = \begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$

(b) B + C cannot be calculated because the matrices are of a different order.

(c) G − E cannot be calculated because the matrices are of a different order.

(d) $C + E = \begin{pmatrix} 12 & 6 & 6 \\ 5 & 5 & 3 \\ 6 & 16 & 15 \end{pmatrix}$

(e) $E - C = \begin{pmatrix} 4 & -2 & -4 \\ -5 & -1 & -1 \\ -6 & 2 & -9 \end{pmatrix}$

(f) $G - D = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

(g) A + E cannot be calculated because the matrices are of a different order.

Quick problem 5

Task One

(a) $2X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(b) $3Y = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

(c) $\frac{1}{2}Z = \begin{pmatrix} \frac{3}{2} & \frac{7}{2} & 3 \\ 2 & \frac{7}{2} & \frac{3}{2} \\ \frac{5}{2} & -1 & \frac{7}{2} \end{pmatrix}$

Task Two
Statement 1 is true since

$$2X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$2(2X) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

and equally

$$4X = 4 \times X = 4 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

Statement 2 is false because the expression cannot be evaluated: the matrices are of a different order.
Statement 3 is false

$$3Z = \begin{pmatrix} 3 \times 3 & 7 \times 3 & 6 \times 3 \\ 4 \times 3 & 7 \times 3 & 3 \times 3 \\ 5 \times 3 & -2 \times 3 & 7 \times 3 \end{pmatrix} = \begin{pmatrix} 9 & 21 & 18 \\ 12 & 21 & 9 \\ 15 & -6 & 21 \end{pmatrix} \neq \begin{pmatrix} 9 & 21 & 18 \\ 12 & 14 & 6 \\ 15 & 6 & 21 \end{pmatrix}$$

Statement 4 is true since

$$5Y = 5 \times \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \end{pmatrix}$$

and

$$\begin{pmatrix} 12 \\ 6 \end{pmatrix} - \begin{pmatrix} 6 \\ 16 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \end{pmatrix}$$

Quick problem 6

(a) $AD = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

(b) $BE = (5 \quad 8 \quad 9 \quad -4)$

(c) AF cannot be found because the rows and columns do not match.

(d) DC cannot be found because the rows and columns do not match.

Quick problem 7

Task One

(a)

$\det(A) = 2$

$\det(B) = 0$

$\det(C) = 23$

(b) Since det(A) and det(C) are both not equal to zero, matrix A and matrix C are non-singular. Since $\det(B) = 0$, matrix B is singular.

Task Two

$$A^{-1} = \begin{pmatrix} 1/2 & 3/2 \\ -1/2 & -1/2 \end{pmatrix}$$

B^{-1} cannot be calculated since B is a singular matrix

$$C^{-1} = \begin{pmatrix} 7/23 & 1/23 \\ 2/23 & 3/23 \end{pmatrix}$$

Quick problem 8

Task One

$p_a = 1$

$p_b = -\frac{2}{3}$

Task Two

p_a value does not make sense: you cannot have a negative price unless the consumer is being paid by the producer to take the product which seems implausible.

Task Three

The values do change and, at the very least, the signs of the figures now make sense.

$p_a = \frac{7}{10}$

$p_b = \frac{1}{10}$

Quick problem 9

Task One

$$A^{-1} = \frac{1}{8}\begin{pmatrix} 7 & -5 & 1 \\ -5 & 7 & -3 \\ -1 & 3 & 1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -3 \\ 1 & 9 & 11 \end{pmatrix}$$

Task Two

The matrix D is a singular matrix and the inverse cannot be calculated. The reason why the matrix is singular is because its determinant is zero.

Quick problem 10

Task One

$P_1 = 5$

$P_2 = -3$, i.e., a loss

$P_3 = 1$

Note change in solutions again the same problem.

Task Two

$P_1 = -1$

$P_2 = 3$

$P_3 = 2$

Quick problem 11

Task One

(a) C = 3, I = −2

(b) The figure for I is negative. Negative investment means that overall the assets of the economy are depreciating without any renewal or improvement. Firms may very pessimistic about the future and so not invest in new plant and machinery.

Task Two

C = 2, I = 1 and G = 3.

Task Three

C = −122/(−94) = 1.30

I = −60/(−94) = 0.64

G = −65/(−94) = 0.69