Lesson I

Applications with Music

Use after Lesson 14-4

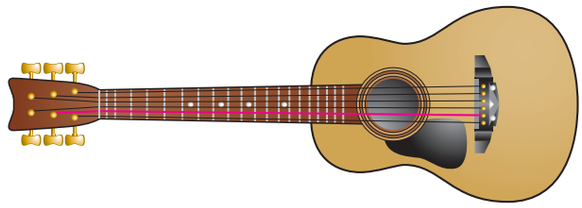
You can use periodic motion, proportions, and geometric transformations to describe mathematical patterns in music.

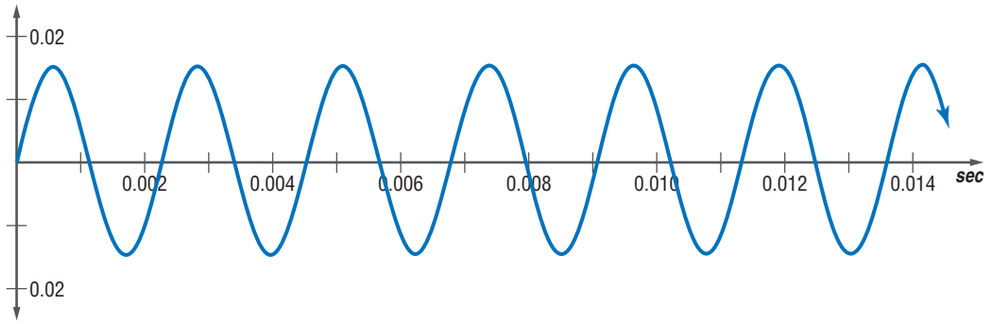
Build Understanding

A musical note is produced by a pattern of vibrations at a frequency that can be heard. When represented on a graph, the **frequency** or pitch of a musical note is the reciprocal of the period, $\frac{1}{T}$, and represents the number of cycles completed per second. One *hertz* (Hz) equals one cycle per second.

Example 1

The sound wave for the note named A above middle C is produced by plucking the string indicated on the guitar shown. This wave of vibrations is modeled by the graph below.



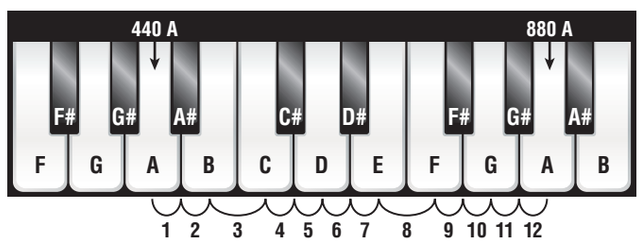


Use this graphical model to estimate the frequency in hertz of A above middle C.

Solution

The sine wave repeats its pattern after between 0.002 and 0.0025 or at about 0.00225 seconds. So $T = 0.00225$ seconds/cycle. The frequency, $\frac{1}{T}$, is then $\frac{1}{0.00225}$ or 444 cycles/second. So A above middle C has a frequency of about 444 Hz.

The exact frequency of A above middle C is 440 Hz. For this reason it is called a 440 A. The A one octave above this note has a frequency of 880 Hz. Notice that there are 12 half steps from 440 A to 880 A. For consecutive pairs of notes, such as notes C and C# or notes E and F, the ratio of their frequencies is constant.



Example 2

Use an algebraic model to find the frequency in Hz of the 2nd D# above middle C, shown in the table at the right.

Note	Frequency (Hz)
A	440
A#	
B	
C	
C#	
D	
D#	?
E	
F	
F#	
G	
G#	
A	880

Solution

Use the frequencies of the notes you know, 440 A and 880 A.

Notice that there are 6 half steps from 440 A to D# and 6 half steps from D# to 880 A. Since D# is located halfway between these two As and there is a constant ratio between the frequencies of each pair of consecutive notes, the ratio of the frequencies of 440 A to D# should be the same as the ratio of the frequencies of D# to 880 A.

Let f represent the frequency of D#. Write and solve a proportion.

first note → $\frac{440}{f} = \frac{f}{880}$ ← first note
next note → next note

$$f \cdot f = 440(880)$$

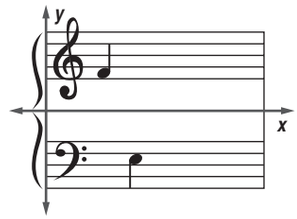
$$f^2 = 387{,}200$$

$$f = \sqrt{387{,}200}$$

$$f \approx 622.25$$

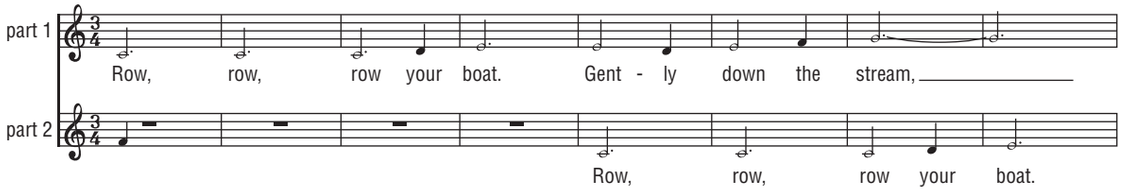
So, the frequency of D# is approximately 622.25 Hz.

Written on sets of five horizontal lines called *staves,* the horizontal position of a musical note represents the time at which it is played, while the note's vertical position represents its pitch. This method of recording notes is similar to using x- and y-axes to graph a relation. You can find patterns similar to geometric transformations in music.



Example 3

Describe the pattern in the portion of music shown below in terms of one or more geometric transformations.

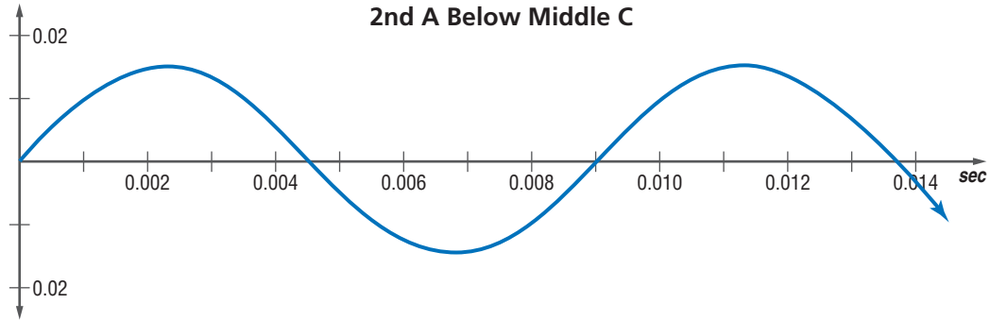


Solution

In this piece of music, one group begins singing "Row, row, row your boat" and then another group begins singing the exact same tune at a different point in time during the piece to produce harmony. This change can be described as a horizontal translation of part 1 to part 2.

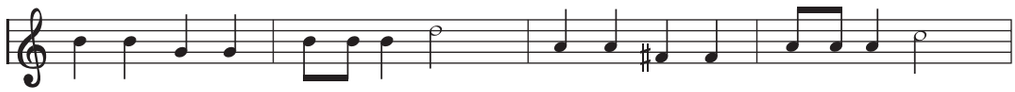
Try These Exercises

1. Use the graphical model below to estimate the frequency in hertz of the indicated note.



2. Refer to the information in Example 2. Use an algebraic model to find the frequency in Hz of F# above middle C.

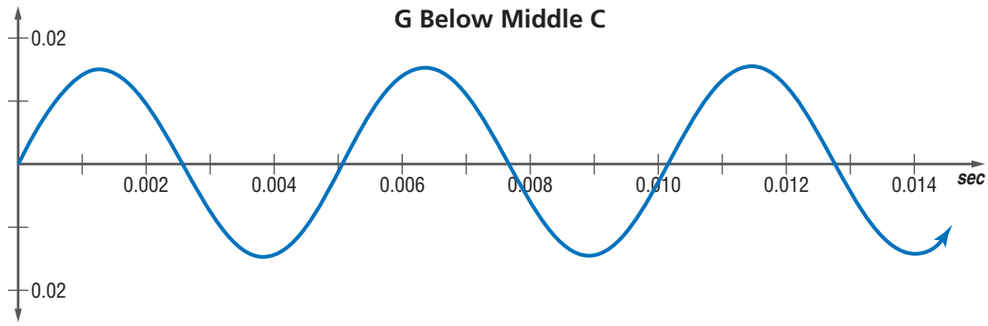
3. Describe the pattern in the portion of music shown below in terms of one or more geometric transformations.



Practice Exercises

Use each graphical model to estimate the frequency in hertz of the indicated note.

4.



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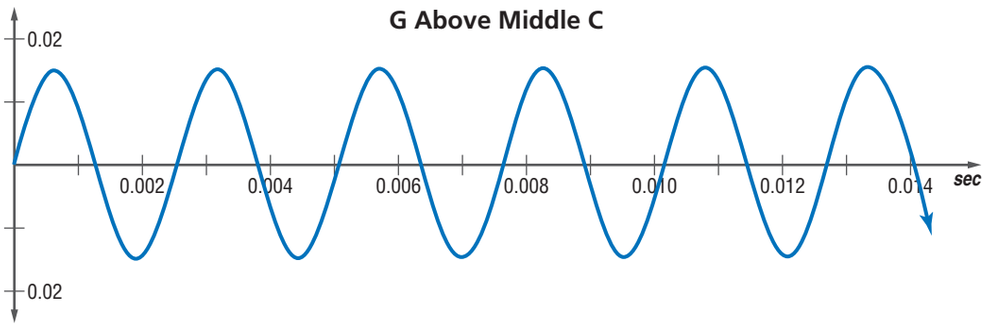

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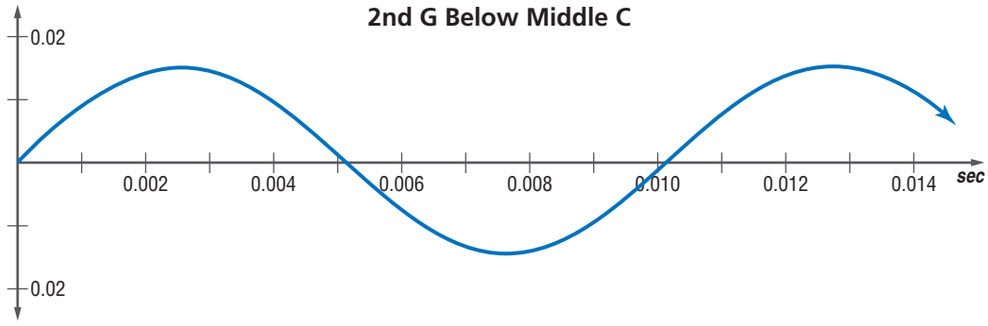


7. Refer to the table in Example 2. Use an algebraic model to find the frequency in Hz of C above middle C.

8. Refer to the information in Example 2 and the answer you found in Exercise 6.

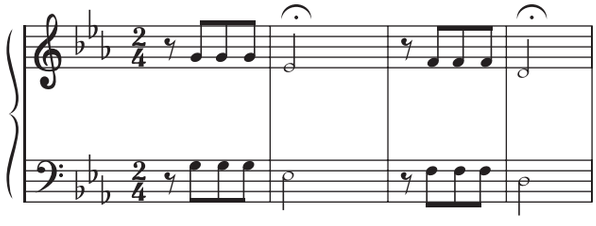
a. How many half steps are there from A above middle C to C above middle C?

b. Find the constant of proportionality p to the nearest hundredth, relating each half step from A above middle C to C above middle C.

c. Use the product of that constant and the frequency for A above middle C to find the frequency of A# above middle C.

Describe the pattern in each portion of music shown in terms of one or more geometric transformations.

9.



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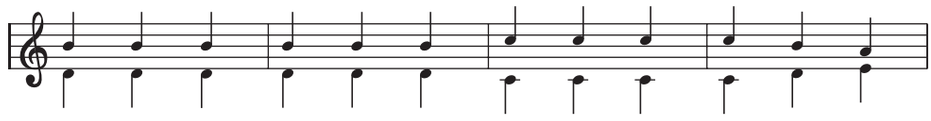
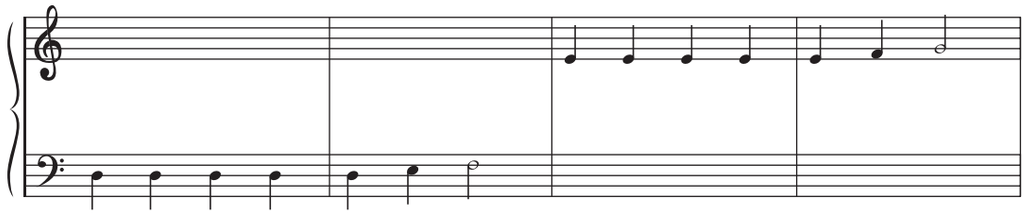


11.



12. **WRITING MATH** Analyze your results from Exercises 7 and 8.

a. If the pitch or frequency of a note is doubled, what affect does this have on the period of the graph of its vibrations?

b. If the pitch of a note is halved, what affect does this have on the period of the graph?