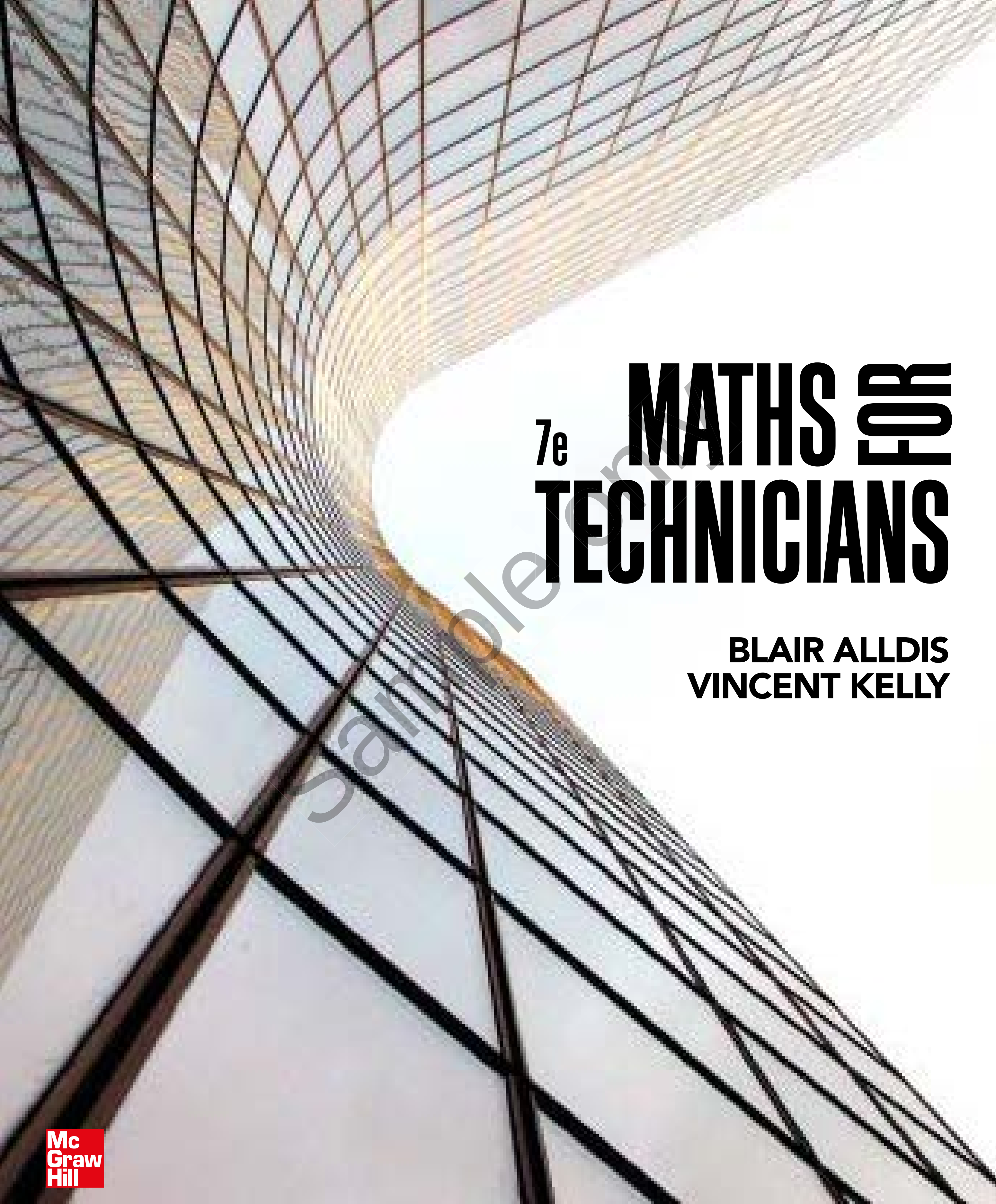

7e

MATHS FOR TECHNICIANS

BLAIR ALLDIS
VINCENT KELLY

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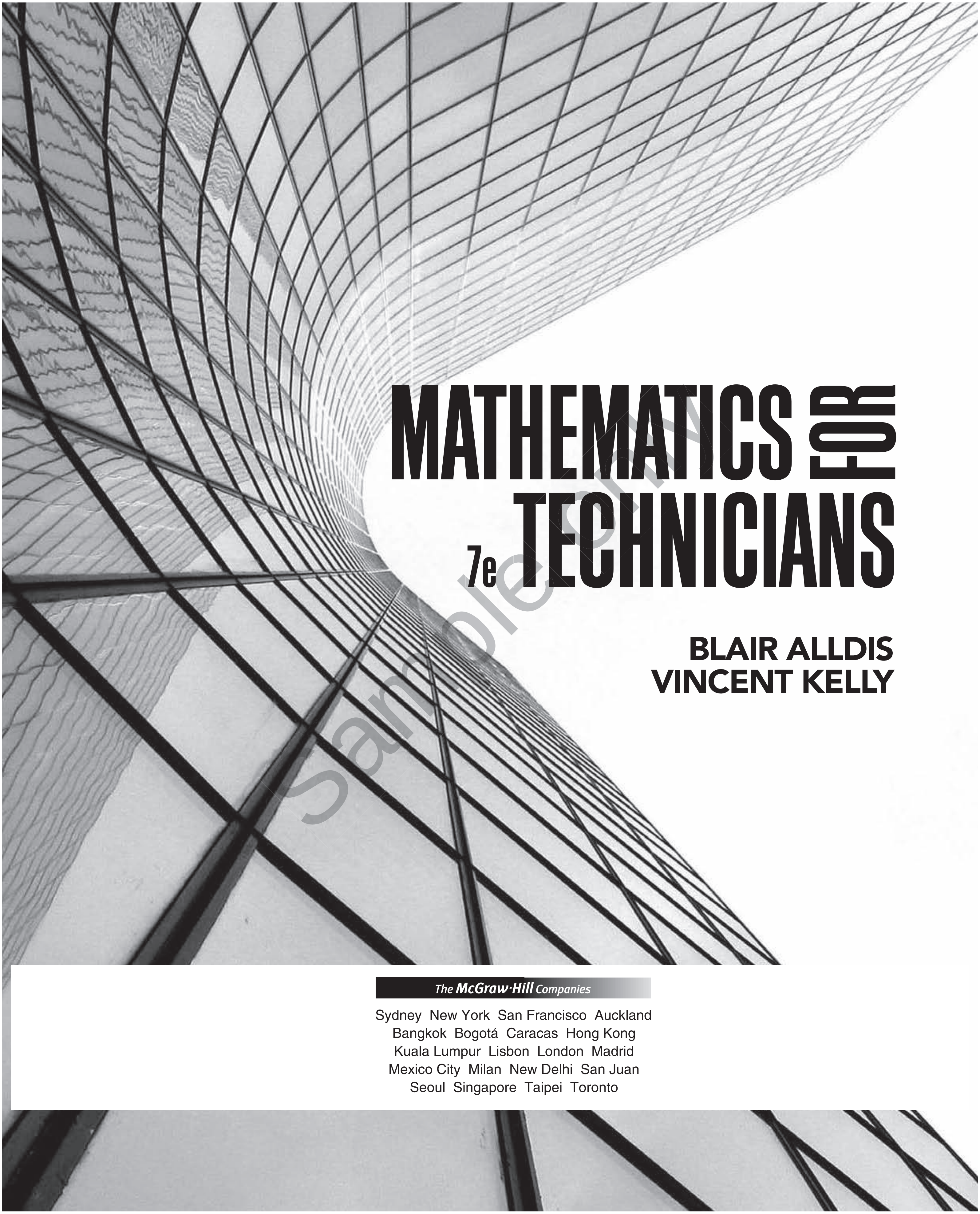

MATHEMATICS FOR TECHNICIANS

7e

BLAIR ALLDIS
VINCENT KELLY

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National Library of Australia Cataloguing-in-Publication Data
Author: Alldis, Blair K. (Blair Knox)
Title: Mathematics for technicians / Blair Alldis, Vincent Kelly.
Edition: 7th ed.
ISBN: 9781743070772 (pbk.)
Notes: Includes index.
Subjects: Shop mathematics—Problems, exercises, etc.
Engineering mathematics—Problems, exercises, etc.
Other Authors/Contributors:
Kelly, Vincent Ambrose.
Dewey Number: 510.2462

Published in Australia by
McGraw-Hill Australia Pty Ltd
Level 2, 82 Waterloo Road, North Ryde NSW 2113
Publisher: Norma Angeloni Tomaras
Senior production editor: Yani Silvana
Copyeditor: Ross Blackwood
Proofreader: Pauline O'Carolan
Indexer: Shelley Barons
Cover and internal design: George Creative
Typeset in Berkeley Book 10.5/14.5pt by Laserwords Private Limited, India
Printed in on by e.g. Printed in China on 70gsm matt art by iBook Printing Ltd

CONTENTS

PART 3 APPLIED MATHEMATICS

ABOUT THE AUTHORS

Vincent Kelly, BEng (Civil), is currently Senior Civil Engineer at Hydro Tasmania. He is an industry expert with more than 15 years' engineering experience and brings this strong practical knowledge to the text. His particular areas of interest are building design and renewable energy.

Blair Alldis BA, BSc, the original author of the previous six successful editions of *Mathematics for Technicians,* has co-authored this seventh edition with Vincent Kelly. Blair has had extensive experience in teaching mathematics to engineering students at TAFE and was Senior Head Teacher of Mathematics at Sydney Institute of TAFE, Randwick College, prior to retiring.

ACKNOWLEDGMENTS

Many thanks to Blair Alldis for the privilege of working on this book; to my wife Corin for all her support and assistance; and to my sons Zac and Sam for their never-ending distraction and amusement. Thanks also to the teachers who assisted in the development of this new edition with their responses to the review questionnaire: Barry Russell from Skills Tech Eagle Farm, Queensland; Les Smith from South Western Sydney Institute of TAFE; Granville College; and John Hungerford. Finally, thanks to the team at McGraw-Hill who have been involved in the production of this edition, in particular Norma Angeloni Tomaras (publisher), Lea Dawson (development editor), Yani Silvana (senior production editor) and Ross Blackwood (freelance copyeditor).

Vincent Kelly

PREFACE

Mathematics for Technicians was developed to motivate engineering students to see the relevance of mathematics to their future careers. For this seventh edition, Vincent Kelly uses his extensive practical engineering experience to build on the strengths of this essential and inspiring well-known text. Practical topics and examples will help to equip students with a sound working knowledge of the elementary mathematics they require to succeed in the many engineering projects they will work on.

The author has also aligned the text with the requirements of TAFE engineering subject units. To achieve this, the following changes have been made:

- Chapters 17 and 18 have been revised. Chapter 17 now concentrates on circular functions and Chapter 18 focuses on trigonometric functions and phase angles.
- The material on determinants and matrices in Chapter 23 has been extended to more thoroughly cover the solution of simultaneous linear equations using matrices, including the solution of three equations with three unknowns.
- A new chapter (Chapter 24) on statistics and probability has been added.
- Supplementary material previously supplied on a CD is now available on the Online Learning Centre (OLC) and includes additional exercises.

TEXT AT A GLANCE

Setting a clear agenda

Each chapter starts by clearly defining learning objectives.

Learning objectives

n completion of this chapter you should be able to udge whether your knowledge of the following topics is satisfactory as a basis for this course.

- olve problems involving ratios.
- olve problems involving direct variation proportion , inverse variation and oint variation.
- se percentage to solve problems mentally, manually and using a calculator.
- erform conversions between fractions, decimals and percentages.
- pply your knowledge of ratio, proportion and percentage to solve practical problems.

If you have difficulty with exercises in this chapter, you should consult your teacher before continuing with this course and seek remedial assistance.

Treatment of theory

Rule: When an equation contains fractions, it is usually advisable to clear all fractions first by multiplying both sides by the LCM of the denominators.

Rule: Since in an equation the pronumeral is standing for some definite number (unknown until the equation is solved), we can multiply both sides of an equation by the pronumeral.

Theory is covered clearly throughout each topic. Where appropriate, **mathematical rules**, **advice**, **notes** and **definitions** are highlighted separately for added emphasis.

Worked examples are provided to illustrate the theory and provide a template for students to develop their own skills.

Examples

1 If $A = \ t -$ and $B = \ - t$, express $A - \ B$ in terms of t.

Solution

$$A - \ B = \ t - \ - \ - t$$
$$= t - \ - \ + \ t$$
$$= t -$$

2 If $x = \frac{-k}{}$ and $y = \frac{k -}{}$, express $x - \ y$ in terms of k.

Solution

$$x - \ y = \left(\frac{-k}{}\right) - \left(\frac{k-}{}\right)$$
$$= -k - \ k -$$
$$= -k - \ k +$$
$$= - \ k$$

Exercises 4.8

1 Simplify:

a $5x - \frac{6x - 4}{2}$ b $T - \frac{6T - 9}{3}$ c $1 - \frac{8 + 6R}{2}$

d $R + 3 - \frac{6R - 8}{2}$ e $3V - \frac{2V - 6}{2}$ f $1 - 4t - \frac{8t + 4}{4}$

2 Express as single fractions:

a $\frac{x}{2} - \frac{x + 1}{3}$ b $\frac{3}{4} - \frac{2 - 4E}{3}$ c $\frac{2}{3} - \frac{3C - 1}{6}$

d $\frac{3V}{4} - \frac{2 + 3V}{6}$ e $\frac{5}{6} - \frac{3L + 5}{4}$ f $\frac{4k}{9} - \frac{k - 5}{6}$

Practice makes perfect

To further develop students' skills and reinforce understanding, several levels of testing are provided throughout the text. A set of exercises follows each theory section. The symbol +++ denotes more difficult exercises.

To ensure complete understanding of each chapter, self-test questions that address core concepts are provided at the end of most chapters. These tests are an excellent revision tool for the classroom.

SELF-TEST

1 \$272 is divided between two people so that the amounts they receive are in the ratio 3 : 5. How much does each receive?
2 In a test, a student scores 67 marks out of a maximum possible mark of 82. What is the approximate percentage mark?
3 The frequency of vibration of a stretched wire when plucked varies inversely as its length and directly as the square root of its tension. If a wire that is 1.40 m long vibrates with a frequency of 376 per second (376 Hz), when its tension is 125 N:
 a what will be its frequency if its length is 2.20 m and its tension is 234 N?
 b what will be the approximate percentage increase or decrease in the frequency?
4 Evaluate mentally:
 a 37% of 8 mL **b** 12% of \$23.50
5 A 50 m cable extends by 25 mm under a load of 1 t. What is the percentage increase in its length? (1000 mm = 1 m)
6 The introduction of an extra machine into a workshop increases the production rate from 170 to 204 articles per week. What is the percentage increase in the production rate?

ANSWERS

1 Fractions and decimals

Exercises 1.1

1 a −8 **b** 4 **c** −2 **d** −2 **e** −4 **f** 2 **2 a** −3 **b** 10 **c** −3 **3 a** 0 **b** 0 **c** −2 **4 a** −2°C **b** −6°C **5 a** contraction of 3 mm **b** contraction of 5 mm **6 a** loss of 2 dB **b** loss of 8 dB **c** gain of 4 dB **7 a** 2 **b** 14 **8 a** −6 **b** −8 **c** −4 **d** 4 **e** 6 **f** 12 **9 a** 4 **b** −2 **c** 6 **d** −4 **e** −8 **f** −1

Exercises 1.2

1 a 14 **b** −1 **c** 26 **2 a** 12 **b** 7 **c** 29 **3 a** 10 **b** 5 **4 a** 40 **b** 6 **c** 9 **d** 2

Exercises 1.3

1 a 2, 3, 5 **b** 3, 9 **c** 2, 3, 4 **d** 3, 5, 9 **e** 2, 3, 9 **f** 3, 5 **g** 2, 4 **h** 3, 9 **2 a** $\frac{13}{15}$ **b** $\frac{16}{17}$ **c** $\frac{3}{4}$ **d** $\frac{7}{19}$ **e** $\frac{3}{4}$ **f** $\frac{4}{7}$ **g** $\frac{2}{3}$ **h** $\frac{3}{5}$ **3 a** $\frac{8}{13}$ **b** $\frac{3}{5}$ **c** $\frac{997}{7000}$ **d** 40

d 1.8 **e** 0.6 **f** 0.000 06 **8 a** 0.09 **b** 0.0004 **c** 0.36 **d** 1.44 **e** 0.0144 **f** 0.000 121 **g** 0.0009 **h** 1.21 **9 a** 0.2 **b** 0.9 **c** 0.07 **d** 1.2 **10 a** 26.08 **b** 1.203 **c** 230.7 **d** 0.0521 **11 a** 2.03 **b** 732.8 **c** 6.003 **d** 1.703 **e** 0.0732 **f** 0.0039 **12 a** 0.68 **b** 5.33 **c** 62.15 **d** 234.20 **e** 0.04 **f** 0.00 **g** 0.01 **h** 0.00 **13 a** 400 **b** 0.567 **c** 0.234 **14 a** 30 **b** 0.4 **c** 0.2 **d** 0.03 **15 a** 20 **b** 20 **c** 0.3 **16 a** 3.6 Ω **b** 200 m **17 a** 0.12 Ω **b** 0.16 Ω **c** 0.024 Ω **d** 0.48 Ω **18 a** 150 **b** 0.06 s **19 a** 1.46 mm **b** 43.8 mm **c** 58.4 mm **d** 7.3 mm **20** 3066 mm^2 **21** 3.4 m/s **22** 4000 W **23** 19.8 J **24** 20 s **25 a** 1.5 s **b** 45 s **26** 4.8 min **27 a** 7000 kg **b** 0.002 m^3

Exercises 1.5

1 a 2500 **b** 2400 **c** 2500 **d** 600 **e** 700 **f** 800

Answers to all exercises and self-test questions are provided at the back of the book and online, enabling self-paced learning.

E-STUDENT / E-INSTRUCTOR

E-student

Online Learning Centre

The Online Learning Centre (OLC) that accompanies this text helps you get the most from your course. It provides a powerful learning experience beyond the printed page.
www.mhhe.com/au/alldis7e

Supplementary material

Supplementary material is now included on the OLC. It provides answers to the questions within the text, as well as additional exercises and theories on a wide range of content. Within the text, the OLC symbol Online Learning Centre indicates that supplementary material is available to enhance your learning experience.

E-instructor

Instructors have additional password-protected access to an instructor-specific area, which includes the following resources.

Teaching notes

To assist in classroom preparation, instructors will have access to teaching notes on selected chapters.

EZ Test Online

EZ Test Online is a powerful and easy-to-use test generator for creating paper or digital tests. It allows easy 'one click' export to course management systems such as Blackboard and straightforward integration with Moodle.

Testbank

A bank of test questions written specifically for this text allows instructors to build examinations and assessments quickly and easily. The testbank is available in a range of flexible formats: in Microsoft Word®, in EZ Test Online or formatted for delivery via Blackboard or other learning-management systems.

CHAPTER 16

COMPOUND INTEREST: EXPONENTIAL GROWTH AND DECAY

Learning objectives

- Solve compound interest problems.
- Draw graphs relating to exponential growth and decay.
- Solve exponential growth and decay problems.

16.1 COMPOUND INTEREST

Online *Learning*Centre

To understand the material in this chapter, you need to apply the knowledge from topics covered in chapter 2 (Section 2.6 Percentages) and chapter 14 (Section 14.9 Exponential equations).

If a number N is increased by r%, it becomes $N + \left(\frac{r}{100} \times N\right) = N \times \left(1 + \frac{r}{100}\right)$.

If this *result* is increased by r%, it becomes $N \times \left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right)$.

If this 'compounding' process is performed n times, N increases to:

$$N \times \left[\left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right) \times \ldots n \text{ terms}\right] = N \times \left(1 + \frac{r}{100}\right)^n.$$

If money is invested in such a way that after certain regular time periods the interest is calculated and added to the account so that *it* attracts interest during the next period, the interest is said to be 'compounded'.

If \$$P$ (the 'principal') is invested for n periods at a compound interest rate of r% per period, there will be n interest payments into the account, each of r% of the amount in the account at that time. The amount to which the money grows in these n periods is given by:

Rule: $A = P \times \left(1 + \frac{r}{100}\right)^n$

This is the 'compound interest formula' where:

- P is the principal sum invested
- $r\%$ is the interest rate over the period of time between interest payments
- A is the amount to which the principal grows after a time interval of n of the above periods.

Example

Calculate the amount that \$5000 grows to when it is invested for three weeks at an interest rate of 13% per year, the interest being compounded daily.

Solution

Interest rate = 13% per year = $\frac{13}{365}$% per day, and there will be 21 interest payments into the account.

$$\therefore\ A = \$5000 \times \left(1 + \frac{\frac{13}{365}}{100}\right)^{21} \approx \$5037.53$$

Exercises 16.1

1 Money is invested at an interest rate of 6% p.a. (per year), the interest being paid and compounded at the end of each year.

a By what percentage does the investment increase in:

i 3 years? **ii** 8 years?

b For how many complete years must the money remain invested before it has more than doubled?

2 The sum of \$2550 is invested at an interest rate of 0.7% per month, the interest being paid and compounded at the end of each month.

a State the *exact* amount on deposit at the end of 5 months and also its value correct to the nearest cent.

b By what percentage will the investment increase in 1 year? State the *exact* percentage and also its value correct to 3 significant figures.

+++ **3** \$3000 is invested for 26 weeks at an interest rate of 7% p.a., the interest being compounded daily. How much total interest is paid on this investment?

16.2 EXPONENTIAL GROWTH

Online LearningCentre

The more frequently the interest is calculated and compounded, the more rapidly the amount in the account grows.

Example

Calculate the amount that $10 000 grows to when it is deposited for 3 years at an interest rate of 5% per year.

Solution

The amount A to which the principal grows in the three years is given by:

- if interest is compounded yearly: $A = \$10\,000 \times \left(1 + \frac{5}{100}\right)^{3} = \$11\,576.25$
- if interest is compounded monthly: $A = \$10\,000 \times \left(1 + \frac{\frac{5}{12}}{100}\right)^{36} = \$11\,614.72$
- if interest is compounded weekly: $A = \$10\,000 \times \left(1 + \frac{\frac{5}{52}}{100}\right)^{3\times52} = \$11\,617.51$
- if interest is compounded daily: $A = \$10\,000 \times \left(1 + \frac{\frac{5}{365}}{100}\right)^{3\times365} = \$11\,618.22.$

It can be shown that as n becomes larger and larger (interest compounded each hour, minute, second, . . .), the amount A does not increase *indefinitely* (i.e. 'without bound') but approaches a limit, which is given by

Rule: $A = P \times e^{kt}$

where
- k is the rate of interest per year expressed as a decimal
- t is the investment period in years
- e is a constant named after the Swiss mathematician Leonard Euler.

Note: The constant e is an irrational number—that is, a non-recurring, non-repeating decimal (like π). Its approximate value is 2.718. This number is so important in technology that, like π, all scientific calculators provide an e^x key. You should check on your calculator that e^1 (= e) ≈ 2.718. You should be able to obtain, for example, $e^3 \approx 20.09$ and $e^{-5} \approx 6.738 \times 10^{-3}$.

Note: Students who are interested should study Appendix A, which contains more information concerning the constant e and the derivation of the above formula for exponential growth.

You can now find the limit to which the amount in the previous example approaches—the amount if the interest is paid *continuously:*

$$A = P \times e^{kt} = \$10\,000 \times e^{0.05\times3} = \$10\,000 \times e^{0.15} = \$11\,618.34$$

Regardless of how often the interest is compounded, the total amount can never exceed this amount. This is the amount if the interest is paid *continuously.* Banks pay interest on some accounts daily or continuously and in both cases the continuous formula would be used since there is very little difference between the amounts in each case.

Exercises 16.2

1 If $P is invested at a compound interest rate of 8% p.a., state the *exact* amount of money on deposit at the end of 1 year:
 a if the interest is paid at the end of each year
 b if the interest is paid quarterly
 c if the interest is paid continuously

2 If $10 000 is invested at an interest rate of 4% p.a., what will this amount become at the end of 1 year, to the nearest cent:
 a if interest is paid each year?
 b if interest is paid each quarter?
 c if interest is paid each month?
 d if interest is paid each week?
 e if interest is paid each day?
 f if interest is paid continuously?

Actually *all* growth must be in 'spurts' but these are often so small and so frequent (e.g. molecule by molecule, quantum by quantum, cent by cent) that the growth may be *considered* to be continuous.

Note: The above formula for exponential growth applies, of course, not only to a quantity of money but to any quantity that grows in this continuous manner where the percentage rate of growth is constant and hence the actual rate of growth is proportional to the amount present at any instant. With a bank deposit invested at compound interest paid continuously, the percentage rate of growth remains constant and so the actual rate of growth (e.g. in dollars per day) increases in proportion to the amount of money in the account.

This type of growth is quite common, an example being population growth (e.g. bacterial, animal or human). When the population of rabbits doubles, the rate of breeding (e.g. in rabbits per second) will double; the rate of growth of the population is proportional to the size of the population at every particular instant. The growth of the population can be regarded as continuous only when the population is large. When the population is small (as was the original population of rabbits in Australia), the growth could not be regarded as continuous. All runaway chain reactions are examples of this type of growth, such as occurs in uncontrolled nuclear fission. This type of growth is sometimes referred to as 'snowballing' because when a snowball rolls down a snow-covered slope, the rate at which it picks up snow at any instant (e.g. in grams per second) is proportional to the weight of the snowball at that instant.

SUMMARY

- Exponential growth is defined as continuous growth in which the actual rate of growth at any instant is proportional to the quantity present at that instant.
- The formula for exponential growth is $Q = Q_0\, e^{kt}$,
  where:
  - Q_0 is the quantity originally present (i.e. the value of Q when $t = 0$)
  - k is the percentage continuous rate of growth during a specified period of time, expressed as a decimal
  - Q is the quantity present at the end of t of the above time periods.

Example

If the mass of a culture increases exponentially from an original mass of 3.45 g at a continuous rate of growth of 13% per hour, find the mass present after two days of growth.

Solution

$$k = 13\% = 0.13 \text{ per hour}$$
$$t = 2 \times 24 = 48 \text{ hours}$$
$$Q = Q_0 e^{kt}$$
$$= 3.45 \times e^{0.13 \times 48}$$
$$= 3.45 \times e^{6.24}$$
$$\approx 1770 \text{ g}$$

We could equally well have used 1 *day* as the time period, in which case $k = 0.13 \times 24 = 3.12$ per day, and $t = 2$ days and $e^{kt} = e^{6.24}$ as before.

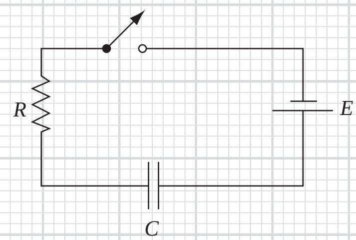
Note: When there is an exponential decrease ('decay') the value of k is negative.

Exercises 16.3

1 The radioactive element radon has a half-life of 3.82 days. (The *half-life* is the time taken for half of any mass to disintegrate.)
Find:
 a the value of k, the continuous fractional rate of growth
 b how long it will take for 100 g of radon to decay to 45.0 g.

2 The population of a particular town, which is currently 10 400, is estimated to be increasing at a rate of 7% per year. Assuming that this rate of growth continues:
 a what is the estimate of the population in 5 years' time?
 b how long will it take for the population to double?
 c how long will it take for the population to increase by 35%?

3 When the key in this circuit is closed, the current decreases exponentially according to the formula $i = \frac{E}{R} \times e^{-\frac{1}{RC}t}$ A/s.



If $E = 180$ V, $C = 120$ μF and $R = 3.8$ kΩ, find the current:
 a immediately after the key is closed
 b 730 ms after the key is closed.

4 A manufacturer of yeast finds that their culture grows exponentially at the rate of 13% per hour.
 a What mass will 3.7 g of yeast grow to:
  i in 7.0 hours? **ii** in two days?

continued

continued

b What mass must be present initially in order to acquire a mass of 15 kg at the end of two days?

c How long will it take for any given mass of this yeast to double?

+++ 5 In the inversion of raw sugar, at any particular moment the rate of decrease of the mass of sugar present is proportional to the mass of sugar remaining. (Remember that this means that the rate of decrease is *exponential*.) If after 8 hours 53 kg has reduced to 37 kg, how much sugar will remain after a *further* 24 hours?

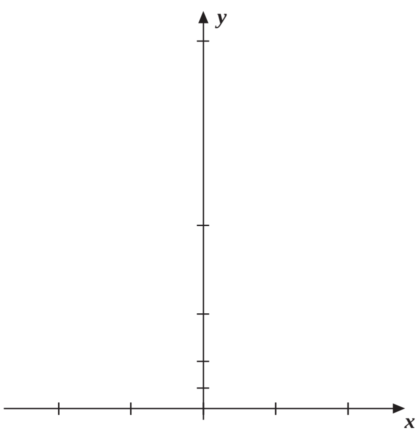
16.3 GRAPHS OF EXPONENTIAL FUNCTIONS

Functions in which the variable appears as an exponent (index) are called 'exponential' functions. Examples are 3.2^x, 7.15^{-3x}, $2.86^{4.31x-2.45}$. The general expression for an exponential function is $K \times a^{bx+c}$, where each of the numbers K, a, b and c may be positive or negative and c may be zero. **A very important property of these functions is that equal increases in the value of x result in equal *percentage* increases in the value of the function.**

$$N^{x+k} = N^x + N^k$$
$$N^{x+2k} = N^{x+k} + N^k$$
$$N^{x+3k} = N^{x+2k} + N^k$$
$$N^{x+4k} = N^{x+3k} + N^k$$

Successive additions of k to the exponent result in a succession of multiplications by the same number, N^k. Hence, as shown in chapter 2 (section 2.6, Percentages) successive additions of k to the exponent result in a succession of increases by the same percentage.

Consider the function $y = K \times a^{bx+c}$. If x is increased by t, the value of y becomes $K \times a^{b(x+t)+c} = K \times a^{bx+c+bt} = K \times a^{bx+c} \times a^{bt}$. Hence the value of y is doubled when $a^{bt} = 2$, i.e. when $bt \log a = \log 2$, or $t = \dfrac{\log 2}{b \log a}$. There is no need to memorize this result but it will assist you to understand the following work if you can follow the reasoning.



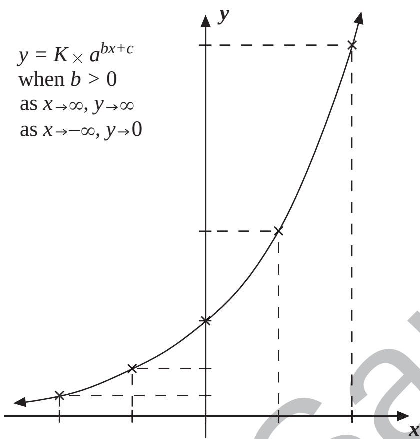
To sketch any curve of the form $y = K \times a^{bx+c}$ when no range of required values is specified for x or y, you can either set up a table of corresponding values of x and y as described in chapter 9 (section 9.6 Dependent and independent variables) or use the following general method, which is much quicker and also provides an understanding of the properties of exponential functions.

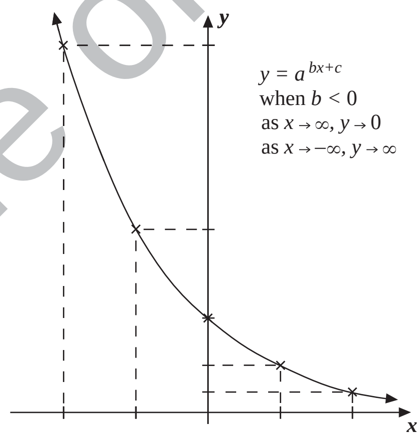
1 Draw axes as shown, placing 2 equally spaced marks on the x-axis on each side of the origin and place 5 marks on the y-axis starting with the top one whose position is the highest you wish your graph to be and each of the succeeding marks below being at half the remaining distance down to the origin. This results in 5 marks, each one being twice the distance from the origin as the mark below it.

2 Calculate the value of y when $x = 0$ and label this value on the middle of your 5 marks on the y-axis. Then mark the values on the other 4 marks on the y-axis, each value being double the value on the mark below it.

3 Calculate the amount by which x must increase in order to double any y-value. Place this value next to the first mark to the right of the origin on the x-axis, and place corresponding values next to the other marks.

4 Mark 5 points on the curve as shown on the figure above, and sketch the curve. To decide which of the two curves below to draw, examine what happens to the value of y as $x \to \infty$. This will depend on whether the exponent b is positive or negative.





All the above should become clear if you study the examples below. Note that in each case the axes are drawn first, then the scale-marks made on each axis as described above.

Example

Sketch the curve $y = 3.2^x$

- As $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to 0$
- When $x = 0$, $y = 1$
- y doubles when 3.2^x increases from 1 to 2.

  If $3.2^x = 2$

  then $\log 3.2^x = \log 2$

continued

continued

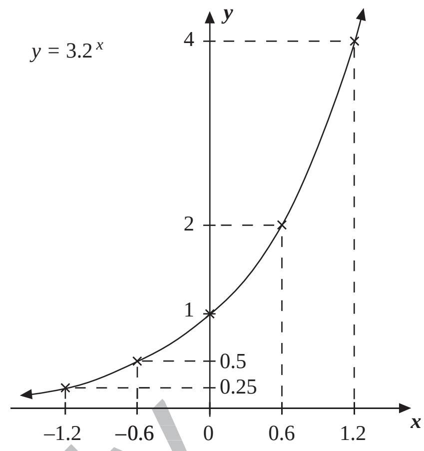
$\therefore x \log 3.2 = \log 2$

$\therefore x = (\log 2) \div (\log 3.2)$

$\therefore x \approx 0.6$

We now have the scales for both axes:

- On the x-axis the equally-spaced values 0.6 units apart.
- On the y-axis the value doubles with each increase of 0.6 on the x-axis.
- These 5 points are sufficient to show the shape of the curve: (−1.2, 0.25), (−0.6, 0.5), (0, 1), (0.6, 2), (1.2, 4).



Example

Sketch the curve $y = 18^{3x+2}$.

- As $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to 0$.
- When $x = 0$, $y = 18^2 = 324$.
- y doubles when $\log 18^{3x+2} = 2 \times 324 = 648$

$\therefore \log 18^{3x+2} = \log 648$

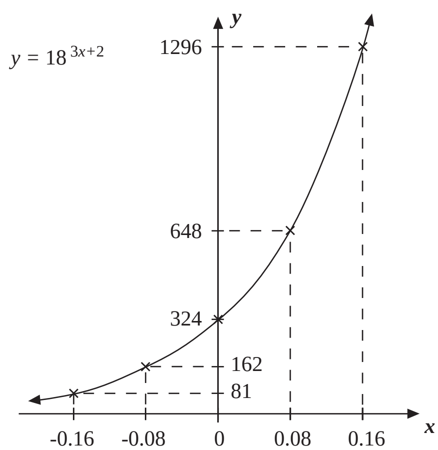
$\therefore (3x + 2) \log 18 = \log 648$

$\therefore 3x + 2 = (\log 648) \div (\log 18) \approx 2.24$

$\therefore x \approx 0.08$

We now have the scales for both axes:

- On the x-axis the equally-spaced values 0.08 units apart.
- On the y-axis the value doubles with each increase of 0.08 on the x-axis.
- These 5 points are sufficient to show the shape of the curve: (−0.16, 81), (−0.08, 162), (0, 324), (0.08, 648), (0.16, 1296).



Example

Sketch the curve $y = 5.9^{2-3x}$.

- As $x \to \infty$, $y \to 0$; as $x \to -\infty$, $y \to \infty$
- When $x = 0$, $y = 5.9^2 \approx 34.8$,
- y doubles when 5.9^{2-3x} increases to 69.6

$\therefore (2 - 3x) \log 5.9 = \log 69.6$

$\therefore 2 - 3x = (\log 69.6) \div (\log 5.9)$

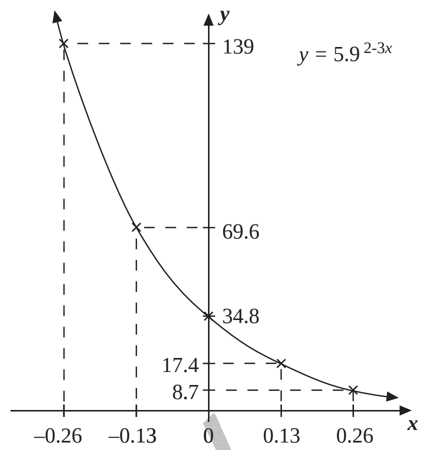
$\therefore 2 - 3x \approx 2.39$

$\therefore 3x \approx -0.39$

$\therefore x \approx -0.13$

We now have the scales for both axes:

- On the x-axis the equally-spaced values 0.13 units apart
- On the y-axis the value doubles with each increase of -0.13 on the x-axis, i.e. with each *decrease* of 0.13.
- These 5 points are sufficient to show the shape of the curve: $(-0.26, 13.9)$, $(-0.18, 69.6)$, $(0, 34.8)$, $(0.13, 17.4)$, $(0.26, 8.7)$.



Exercises 16.4

Sketch each of the following curves. You may use the long method by setting up your own table of values for x and y or the quicker method described in the examples above. The answers provided are the coordinates of the 5 points obtained using the quick method, but if you use the long method the 5 points given in the answers should lie on your curve.

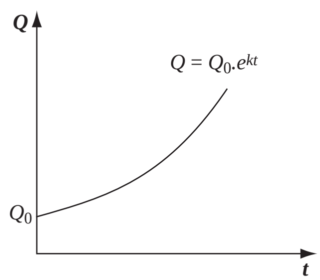
1 $y = 6.3^{1.25x + 0.83}$ **2** $y = 0.78^{4.5x - 5.6}$ **3** $y = 4.3^{2.6 - 2.5x}$

4 $y = 13^{0.9x + 1.3}$ **5** $y = 81^{1.3x - 0.39}$

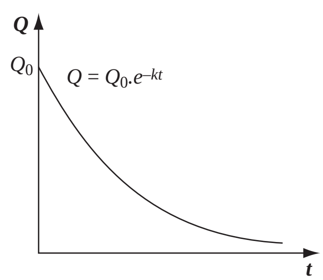
16.4 EXPONENTIAL RELATIONSHIPS

Three very common relationships are:

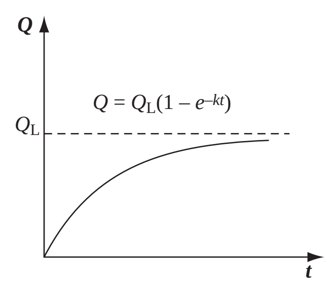
a



b



c



The values of Q, Q_0, t and k were defined in the Summary in section 16.2 above on page 276.

Examples of exponential growth

See the above figure **a** for this example.

- Population growth (human, animal, bacteria cultures—provided nothing such as famine interferes with the rate of growth).
- All 'chain reactions'.

Examples of exponential decay

See the above figure **b** for this example.

- The radioactive decay of a substance, Q being the mass of the substance remaining unchanged at time t.
- A heated body cooling, Q being the excess of the body's temperature above room temperature (*Newton's law of cooling*).
- The speed of a rotating flywheel when the power is disconnected, Q being measured in revolutions per minute, for example.
- Electrical quantities.

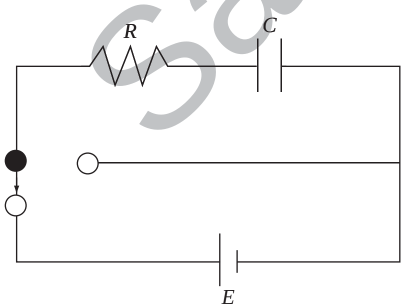
Examples of exponential growth towards an upper limit

See the above figure **c** for this example.

- The concentration of a substance during a chemical reaction, Q_L being the final limiting concentration.
- The temperature of a body after it is placed in an oven, Q being the excess of the body's temperature above its original temperature, and Q_L the temperature of the oven.
- The speed of an electrically driven flywheel after the power is connected.
- Electrical quantities.

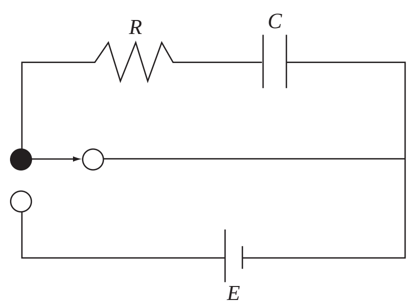
Students of electrical engineering will easily be able to interpret the following diagrams. For a series R–C or a series L-C circuit, when charging or discharging, all the currents and voltages increase or decrease exponentially. (Current = rate of flow of charge.)

Charging

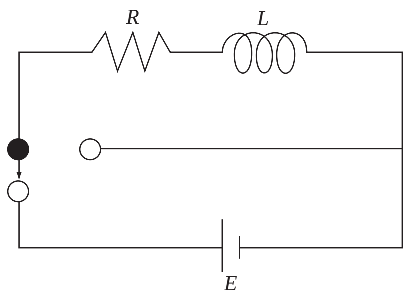


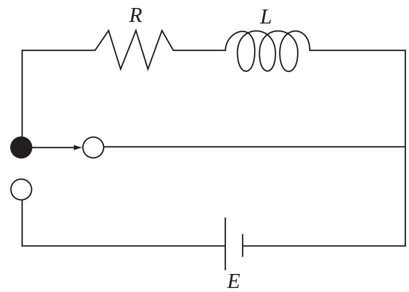
$$i = \frac{E}{R}e^{-\frac{1}{RC}t}$$
$$V_R = iR = Ee^{-\frac{1}{RC}t}$$
$$V_C = E - V_R = E(1 - e^{-\frac{1}{RC}t})$$

Discharging



$$i = \frac{E}{R}e^{-\frac{1}{RC}t}$$
$$V_R = iR = Ee^{-\frac{1}{RC}t}$$
$$V_C = -V_R = -Ee^{\frac{1}{RC}t}$$





$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$
$$V_R = iR = E\left(1 - e^{-\frac{R}{L}t}\right)$$
$$V_L = E - V_R = Ee^{-\frac{R}{L}t}$$

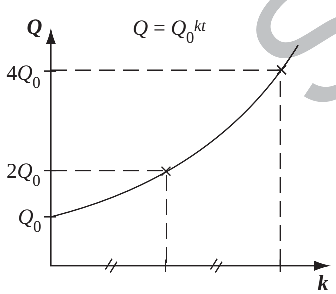
$$i = \frac{E}{R}e^{-\frac{R}{L}t}$$
$$V_R = iR = Ee^{-\frac{R}{L}t}$$
$$V_L = \ -V_R = \ -Ee^{-\frac{R}{L}t}$$

Each of the above 12 equations is of the form $Q = Q_0\, e^{-kt}$ or $Q = Q_L\,(1 - e^{-kt})$, whose graphs are shown at the beginning of this section. In each case, quantity i, V_R, V_C or V_L either commences at an initial value Q_0 and decays towards zero, or commences at value zero and grows exponentially towards a final limiting value, Q_L.

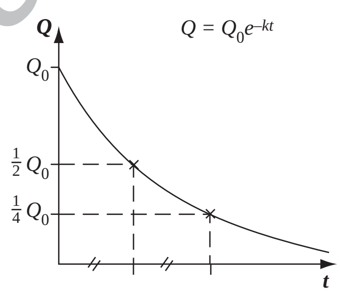
For example, $I = \frac{E}{R}e^{-\frac{1}{RC}t}$ means that the current commences at value $\frac{E}{R}$ (when $t = 0$) and then decays exponentially with time towards zero as $t \to \infty$. In this case the continuous fractional rate of increase of i per unit of time is $k = -\frac{1}{RC}$.

Also, $i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ means that the current commences at value zero (when $t = 0$) and grows exponentially towards a final limiting value of $\frac{E}{R}$ (as $t \to \infty$). In this case the continuous fractional rate of increase of i per unit of time is $k = \frac{R}{L}$.

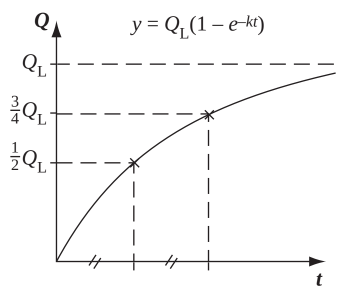
Note how the values of the functions change with equal increases in the value of x:



Exponential growth



Exponential decay



Exponential growth from zero to a limit Q_L

The examples below show the steps in sketching such functions.

Rule: A reminder: $\log_{10} 10 = 1$, $\log_e e = 1$ (i.e. $\ln e = 1$)

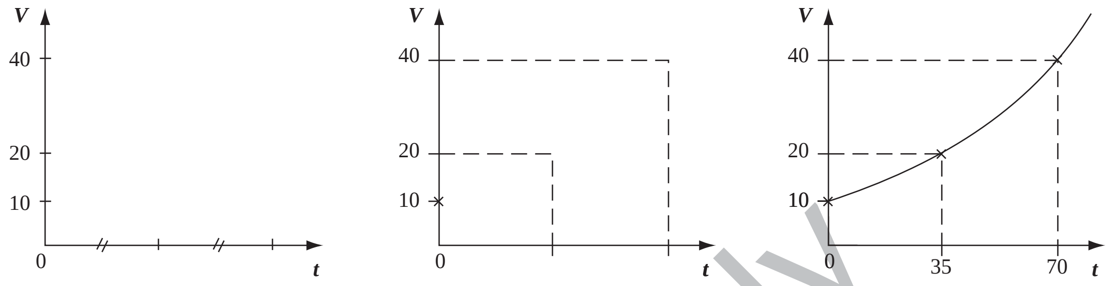
Examples

1 Sketch the graph of $V = 10e^{0.02t}$.

When $t = 0$, $V = 10e^{0.02t} = 10$

V doubles when $10 \times e^{0.02t} = 20$, i.e. $e^{0.02t} = 2$,

i.e. when $0.02t = \ln 2$, or $t \approx 35$



2 Sketch the graph of $M = 400(1 - e^{-1.25t})$.

When $t = 0$, $M = 400\,(1 - 1) = 0$

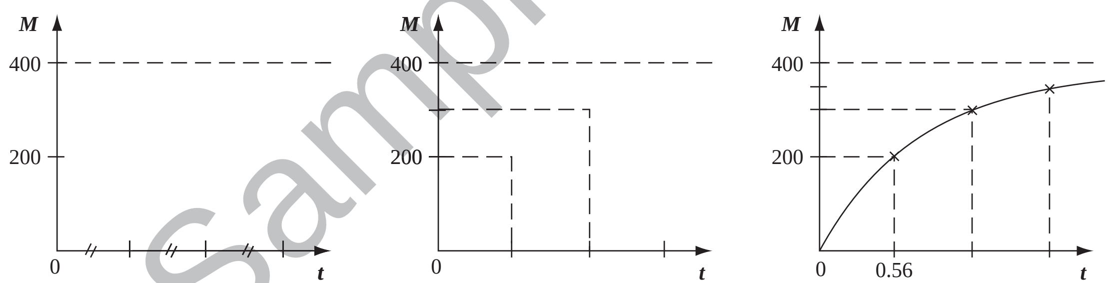
As $t \to \infty$, $M \to 400$

M halves when $400(1 - e^{-1.25t}) = 200$

i.e. when $e^{-1.25t} = 0.5$

i.e. when $-1.25t = \ln 0.5$

or $t \approx 0.56$.



Exercises 16.5

1 Draw freehand, quick sketch-graphs of the functions:

a $y = 2^x$, for $-3 \le x \le 3$

b $y = 2^{-x}$, for $-3 \le x \le 3$

c $y = 1 - 2^{-x}$, for $0 \le x \le 4$

d $y = 327(1 - 2^{-x})$, for $0 \le x \le 4$

2 Draw accurate graphs of the functions:

a $y = 5e^{0.2x}$ for $-6 \le x \le 10$

b $y = 24(1 - e^{-8x})$ for $0 \le x \le 0.5$

AQ: Check +++ **3** **a** Calculate the values of the function $Q = 16(1 - e^{-500t})$ when x has the values $-2, -1, 0, 1, 2$ and 3, and use these values to plot the graph of the function over this range.

b What numbers correct to 2 significant figures would replace the numbers 1, 2 and 3 on the x-axis of the graph in part **a** in order to convert it to the graph of the function:

i $y = 2^{2x}$ ii $y = 16^x$ iii $y = e^x$ iv $y = e^{2x}$?

(Hint: Replace 1 by the number that makes the new function equal to 2.)

c What numbers would replace the numbers 1, 2, 4 and 8 on the y-axis of the graph in part **a** in order to convert it to the graph of the function $y = 43 \times 2^x$?

4 Find the approximate values of x for which:

a $e^x = 2, 4, 8$ and 16 b $e^x = 1, \frac{1}{2}, \frac{1}{4}$ and $\frac{1}{8}$

Hence plot a graph of the function e^x over the above range of values.

5 The number of bacteria in a particular culture is given by $N = 1000 \times e^{2t}$, where the time t is in minutes. State how long it takes:

a for the number to double

b for the number to increase from 1000 to 4000

c for the number to increase from 1000 to 8000.

d Draw a graph showing the growth of the culture from 1000 to 8000. From your graph find approximately how long it takes for the number to increase from 1000 to 6000.

+++ 6 The electric current through an induction coil is given by $I = 200(1 - e^{-0.5t})$ mA, where the time t is in seconds.

a Find the times taken for the current to become 100 mA, 150 mA and 175 mA.

b Draw a graph showing the growth of the current during the first 5 seconds.

c From your graph find the approximate size of the current after 1 second.

SELF-TEST

1 \$7000 is invested for 4 years at an interest rate of 5% p.a. What sum, to the nearest cent, does this grow to:

a if the interest is compounded yearly?

b if the interest is compounded each quarter?

c if the interest is compounded daily?

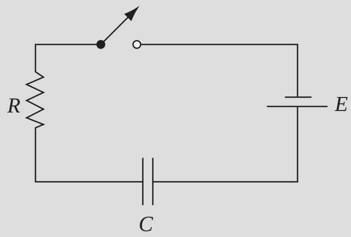
d if the interest is compounded continuously?

2 A certain bacterial culture grows so that each cell (on average) takes 9.70 minutes for mitosis (i.e. the ‘doubling time’—the time for one cell to divide into two cells—is 9.70 minutes).

a Calculate the continuous rate of growth k, correct to 3 significant figures.

b If 3.20 mg of bacteria is originally present, how much will be present after 2 hours’ growth?

+++ 3



When the key is closed, the voltage E across the capacitor C increases exponentially towards a maximum value according to the equation $E = 85(1-e^{-kt})$, where t is in seconds.

a What is the voltage of the source?

b If E rises from 0 to 25 V in the first 730 ms, how long does it take for E to increase from 0 to 80 V?

4 Draw sketch-graphs of the following curves:

a $y = 5e^{14x}$ for $0 \leq x \leq 0.2$

b $i = 40e^{-100t}$ mA, where t is in seconds

c $Q = 16(1-e^{-500t})$ mC, where t is in seconds

5 A radioactive substance decomposes so that the mass present after t years is given by $M = 100e^{-0.1t}$ g.

a Find, to the nearest year, how long it will take for the mass present to reduce to:

i 50 g **ii** 25 g **iii** 12.5 g **iv** 3.125 g

b Draw a graph showing the decay from 100 g to 3.125 g and from your graph find approximately the mass present after 10 years.

CHAPTER 23

DETERMINANTS AND MATRICES

Learning objectives

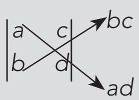
- Evaluate determinants.
- Solve simultaneous equations using determinants.
- Multiply matrices.
- Understand the algebra of matrices.
- Calculate the inverse of a matrix.
- Express simultaneous equations in matrix form.
- Solve simultaneous linear equations using matrices.

There is another method for solving simultaneous linear equations that is applicable to *all* such equations and often saves a lot of time. Once learnt, this method is very concise, very simple to use and provides less opportunity for careless error.

However (there is always a catch, of course), there are a few facts that you will have to learn first.

23.1 DEFINITION AND EVALUATION OF A 2 × 2 DETERMINANT

$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is called a **two-by-two determinant.** It is an array of numbers having two rows and two columns enclosed between vertical bars. It is shorthand notation for $ad - bc$.

Note: 

Examples

1 $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = (2 \times 5) - (4 \times 3) = -2$

2 $\begin{vmatrix} -2 & 3 \\ -4 & -5 \end{vmatrix} = (-2 \times -5) - (-4 \times 3) = 10 + 12 = 22$

Exercises 23.1

Evaluate the following 2 × 2 determinants:

1 $\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$

2 $\begin{vmatrix} 2 & 0 \\ 6 & 1 \end{vmatrix}$

continued

continued

3 $\begin{vmatrix} 2 & 6 \\ 5 & 3 \end{vmatrix}$

4 $\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix}$

5 $\begin{vmatrix} 3 & -1 \\ -4 & -2 \end{vmatrix}$

6 $\begin{vmatrix} 6.72 & 3.91 \\ 4.84 & 5.06 \end{vmatrix}$

23.2 SOLUTION OF SIMULTANEOUS EQUATIONS USING 2 × 2 DETERMINANTS

If we have two simultaneous equations, for example:

$$\begin{cases} 2x + 3y = 4 \\ 5x + 6y = 7 \end{cases}$$

$\triangle$ is the determinant we obtain from the coefficients on the left-hand sides, in the same order and arrangement as they appear in the equations.

In this case, $\triangle = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$

$\triangle_x$ is the same determinant as above, except that the x-coefficients are replaced by the right-hand numbers of the equations.

In this case, $\triangle_x = \begin{vmatrix} 4 & 3 \\ 7 & 6 \end{vmatrix} = 24 - 21 = 3$

$\triangle_y$ is the same determinant as $\triangle$ but with the y-coefficients replaced by the right-hand numbers of the equations.

In this case, $\triangle_y = \begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix} = 14 - 20 = -6$

Examples

1 If $\begin{cases} 3x + 5y = 2 \\ 6x + 4y = 7 \end{cases}$

then $\triangle = \begin{vmatrix} 3 & 5 \\ 6 & 4 \end{vmatrix}$
$= 12 - 30$
$= -18$

$\triangle_x = \begin{vmatrix} 2 & 5 \\ 7 & 4 \end{vmatrix}$
$= 8 - 35$
$= -27$

$\triangle_y = \begin{vmatrix} 3 & 2 \\ 6 & 7 \end{vmatrix}$
$= 21 - 12$
$= 9$

2 If $\begin{cases} 4x - 3y = -5 \\ -2x - y = 6 \end{cases}$

then $\triangle = \begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix}$
$= (-4) - (6)$
$= -4 - 6$
$= -10$

$\triangle_x = \begin{vmatrix} -5 & -3 \\ 6 & -1 \end{vmatrix}$
$= (5) - (-18)$
$= 5 + 18$
$= 23$

$\triangle_y = \begin{vmatrix} 4 & -5 \\ -2 & 6 \end{vmatrix}$
$= 24 - 10$
$= 14$

Exercises 23.2

1 Given that $\begin{cases} 4x + 3y = 1 \\ x + 2y = 5 \end{cases}$

evaluate: **a** $\triangle$ **b** $\triangle_x$ **c** $\triangle_y$

2 Given that $\begin{cases} m - 5t = 3 \\ -2m + 4t = -1 \end{cases}$

evaluate: **a** $\triangle$ **b** $\triangle_m$ **c** $\triangle_t$

3 Given that

$\begin{cases} 3.6V + 2.8e = 1.8 \\ -1.2V - 5.7e = -3.3 \end{cases}$

evaluate: **a** $\triangle$ **b** $\triangle_v$ **c** $\triangle_e$

Solution of simultaneous equations

Consider any two simultaneous linear equations:

$$\begin{cases} a_1x + b_1y = c_1 \quad \text{———— (1)} \\ a_2x + b_2y = c_2 \quad \text{———— (2)} \end{cases}$$

$$\therefore \quad a_1b_2x - b_1b_2y = b_2c_1 \quad \text{———— (1)} \times b_2$$

$$a_2b_1x + b_1b_2y = b_1c_2 \quad \text{———— (2)} \times b_1$$

Subtracting: $a_1b_2x - a_2b_1x = b_2c_1 - b_1c_2$

$$\therefore x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2$$

$$\therefore x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$$

$$= \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$= \frac{\triangle_x}{\triangle}$$

Similarly, it may be shown that $y = \frac{\triangle_y}{\triangle}$.

SUMMARY

The solution of two simultaneous linear equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

is: $x = \frac{\triangle_x}{\triangle}$, $y = \frac{\triangle_y}{\triangle}$

This is known as 'Cramer's Rule'. To apply this rule, both equations must be expressed with the constants c_1 and c_2 on the right-hand sides.

Example

Solve:

$$\begin{cases} 23.5I_2 + 16.7I_1 - 58.7 = 0 \\ 81.2I_1 - 34.2I_2 + 13.9 = 0 \end{cases}$$

We first rearrange the equations: $\begin{cases} 16.7I_1 + 23.5I_2 = 58.7 \\ 81.2I_1 - 34.2I_2 = -13.9 \end{cases}$

$$\triangle = \begin{vmatrix} 16.7 & 23.5 \\ 81.2 & -34.2 \end{vmatrix} = -571.14 - 1908.2 = -2479.34$$

$$\triangle_{I_1} = \begin{vmatrix} 58.7 & 23.5 \\ -13.9 & -34.2 \end{vmatrix} = -2007.54 + 326.65 = -1680.89$$

$$\triangle_{I_2} = \begin{vmatrix} 16.7 & 58.7 \\ 81.2 & -13.9 \end{vmatrix} = -232.13 - 4766.44 = -4998.57$$

$$I_1 = \frac{\triangle_{I_1}}{\triangle} = \frac{-1680.89}{-2479.34} \approx 0.678, \quad I_2 = \frac{\triangle_{I_2}}{\triangle} = \frac{-4998.57}{-2479.34} \approx 2.02$$

Answer: $I_1 \approx 0.678$, $I_2 \approx 2.02$

Note the advantage of this method. Applying the substitution or elimination method would be very tedious for such equations.

Exercises 23.2 *continued*

Solve the following simultaneous equations correct to 3 significant figures:

4 $\begin{cases} 7E + 19V = 24 \\ 13E - 5V = 9 \end{cases}$

5 $\begin{cases} 13L - 15W = 87 \\ 17L - 11W = 231 \end{cases}$

6 $\begin{cases} 2.70x - 1.40y = 3.40 \\ 3.80x + 4.60y = 1.30 \end{cases}$

7 $\begin{cases} 29.3I_1 - 37.8I_2 = 83.7 \\ 41.2I_1 - 26.4I_2 = -16.3 \end{cases}$

23.3 MATRICES: INTRODUCTION

Matrices can be used, among other purposes, to solve simultaneous equations, provided we define their operations (e.g. addition and multiplication) in special ways. The interest in and use of matrices has increased greatly since the introduction of computers because their operations are easy to program on a computer. Manually, for example, it is usually quicker to solve simultaneous equations using determinants. However, with a computer, matrices are much easier to use, regardless of how many variables are involved.

Definitions

A *matrix* is a set of numbers (called *elements*) arranged in a rectangular pattern (or *array*) of rows and columns. A *determinant,* as we saw in section 23.1, is this kind of array distinguished by a vertical bar at each side. To distinguish a *matrix,* the array is enclosed in *parentheses,* either round or square.

For example, $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$ is a *determinant*, which has the value 5, but $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ is a *matrix*, which does not have a value. **A matrix does not represent a number.**

If a matrix has a rows and b columns, it is said to be an '$a \times b$ matrix' or to 'have an *order* of $a \times b$'. A matrix of order '$a \times 1$' is called a *column matrix*. A matrix of order '$1 \times b$' is called a *row matrix*.

Examples

1 $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ -2 & 4 \\ 5 & 3 \end{pmatrix}$ is a matrix of order 3×2.

2 $\mathbf{D} = \begin{pmatrix} 5 \\ 3 \\ -1 \\ 2 \end{pmatrix}$ is a matrix of order 4×1, a *column matrix*.

3 $\mathbf{K} = \begin{pmatrix} 5 & -3 & 2 \end{pmatrix}$ is a matrix of order 1×3, a *row* matrix.

4 $\mathbf{P} = \begin{pmatrix} 4 & -2 \\ 0 & 3 \end{pmatrix}$ is a 2×2 matrix, a *square* matrix of order 2.

5 $\mathbf{T} = \begin{pmatrix} -1 & 0 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 0 \end{pmatrix}$ is a 3×3 matrix, a *square* matrix of order 3.

We identify a matrix by a capital letter and its order can be shown under this letter. For example, $\underset{3\times 2}{\mathbf{K}}$ is a matrix that we call **K** and its order is 3×2 (i.e. 3 rows and 2 columns).

Exercises 23.3

Below is a set of matrices that are also referred to in following exercises.

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ $\quad \mathbf{C} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$\mathbf{D} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$ $\quad \mathbf{E} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 2 \\ 4 & 1 & 3 \end{pmatrix}$ $\quad \mathbf{F} = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$

$\mathbf{G} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}$ $\quad \mathbf{H} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ $\quad \mathbf{K} = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$

1 Using the above set of matrices, state the order of: **a D**, **b G**, **c C**, **d A**, **e F**.

2 Which of the matrices in the above set is: **a** a 2×3 matrix, **b** a 3×2 matrix, **c** a square matrix, **d** a row matrix, **e** a column matrix?

23.4 SOME DEFINITIONS AND LAWS

Equal matrices

Two matrices are said to be *equal* if, and only if, they are identical in every respect—that is, the elements of each are the same and in the same positions.

Example

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$, then $a = 2, b = 4, c = 1, d = 3$.

The sum or difference of two matrices

These are found by adding or subtracting the corresponding elements of each matrix.

Example

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix} = \begin{pmatrix} a+g & b+h & c+i \\ d+j & e+k & f+l \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 0 \\ 5 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -3 & 3 & -2 \end{pmatrix}$$

Note: Two matrices can be added or subtracted only when they have the same order (i.e. they must have the same number of rows and the same number of columns; they must have the 'same shape'). The resulting sum or difference will also have the same order.

From the definition, it can be seen that **A + B = B + A** (the commutative law for addition).

The zero matrix

For matrix **A** the zero matrix, **O**, is defined to be the matrix such that **A + O = O + A = A** (the law of addition of zero). The letter **O** is used to denote a zero matrix.

Example

The zero matrix of $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The zero matrix of $\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Note: The matrix **O** is not the *number* zero but is the matrix of the same order as **A**, which has the number 0 for each of its elements.

Multiplication by a constant

By definition, a matrix is multiplied by a constant by multiplying *every element* of the matrix by that constant.

Example

$$3 \times \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -3 & 0 \\ 0 & 9 & -6 \end{pmatrix}$$

Exercises 23.4

1 Solve the following matrix equations:

a $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} x + 2 \\ y - 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

c $\begin{pmatrix} x - 1 \\ 2y + 3 \end{pmatrix} = \begin{pmatrix} 3 - x \\ y + 9 \end{pmatrix}$

d $\begin{pmatrix} x + y \\ x - y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

2 Write the single matrix equation

$$\begin{pmatrix} 3x + 2y - 5z \\ 4x - 3y + 2z \\ 5x + 5y - 3z \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 9 \end{pmatrix}$$

as three separate simultaneous equations.

3 Using the set of matrices (**A**, **B**, **C**, . . ., **K**) in Exercises 23.3:

a state which pairs of those matrices can be added or subtracted

b write down the matrix **K** + **E**

c state the zero matrix of **D**

d state the zero matrix of **E**

e write down the matrix 3 × **A**

f write down the matrix 2**E** + 3**K**

23.5 MULTIPLICATION OF MATRICES

Since a matrix is simply an array (arrangement) of numbers in a rectangular pattern and does not have a value, we can define the product of two matrices in any way we choose.

For the purposes of explanation, the rows and columns of a matrix will be designated as shown in the matrix below: the rows being called R_1, R_2, R_3, . . . and the columns C_1, C_2, C_3, ...

Any particular element of a matrix can be identified by stating its row and its column.

$$\begin{matrix} & C_1 & C_2 & C_3 \\ & \downarrow & \downarrow & \downarrow \end{matrix}$$

For example, in the matrix $\begin{matrix} R_1 \rightarrow \\ R_2 \rightarrow \end{matrix} \begin{pmatrix} 3 & 6 & 7 \\ 2 & 5 & 4 \end{pmatrix}$, $R_1C_3 = 7$, $R_1C_1 = 3$ and $R_2C_2 = 5$.

By definition, when two matrices are multiplied, the product is another matrix and regardless of how many rows and columns the matrices may possess, when the elements in R_n of the first matrix are multiplied in succession by the elements of C_m of the second matrix and these products are added, this gives element R_nC_m of the product matrix.

When put into words this definition seems very complicated but some illustrations and some practice should enable you to gain facility with this process.

Ignoring all the other rows and columns that may be present:

1 $$R_3 \rightarrow \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 5 & 2 & 7 \\ \times & \times & \times \end{pmatrix} \times \begin{pmatrix} \times & \overset{\substack{C_2 \\ \downarrow}}{4} & \times \\ \times & 0 & \times \\ \times & 1 & \times \end{pmatrix} = R_3 \rightarrow \begin{pmatrix} \times & \overset{\substack{C_2 \\ \downarrow}}{\times} & \times \\ \times & \times & \times \\ \times & 27 & \times \\ \times & \times & \times \end{pmatrix}$$

In the product matrix, element $R_3C_2 = (5 \times 4) + (2 \times 0) + (7 \times 1) = 27$

2 $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 & 7 & 9 \\ 6 & 8 & 10 \end{pmatrix} = \begin{pmatrix} (1 \times 5) + (2 \times 6) & (1 \times 7) + (2 \times 8) & (1 \times 9) + (2 \times 10) \\ (3 \times 5) \times (4 \times 6) & (3 \times 7) + (4 \times 8) & (3 \times 9) + (4 \times 10) \end{pmatrix}$$

$$= \begin{pmatrix} 5 + 12 & 7 + 16 & 9 + 20 \\ 15 + 24 & 21 + 32 & 27 + 40 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 23 & 29 \\ 39 & 53 & 67 \end{pmatrix}$$

You are advised to practise this process until it becomes quite familiar to you. Below are some exercises to enable you to practise the multiplication of two matrices. You will quickly discover the benefit of using fingers or a pen to obscure the rows and columns not being used to obtain a particular element of the product matrix.

Example

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} (2 \times 4) + (3 \times 1) & (2 \times 2) + (3 \times 3) \\ (4 \times 4) + (5 \times 1) & (4 \times 2) + (5 \times 3) \end{pmatrix}$$

$$= \begin{pmatrix} 8 + 3 & 4 + 9 \\ 16 + 5 & 8 + 15 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 13 \\ 21 & 23 \end{pmatrix}$$

When the elements are small you should be able to obtain the product matrix without needing to write down the intermediate steps.

e.g. $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 8 & 12 \end{pmatrix}$

Exercises 23.5

1 **a** You are given the following matrices:

A $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, **B** $\begin{pmatrix} 5 & 1 \\ 3 & 2 \end{pmatrix}$, **C** $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, **D** $\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$, **E** $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Write down the following matrices:

i **A** × **B** **ii** **A** × **C** **iii** **A** × **D** **iv** **A** × **E**
v **B** × **C** **vi** **B** × **D** **vii** **B** × **E** **viii** **C** × **D**
ix **C** × **E** **x** **D** × **E**

b You are given the following matrices:

P $\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$, **Q** $\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$, **R** $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, **S** $\begin{pmatrix} -2 & 0 \\ -3 & 0 \end{pmatrix}$, **T** $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$

Write down the following matrices:

i **P** × **Q** **ii** **P** × **R** **iii** **P** × **S** **iv** **P** × **T**
v **Q** × **R** **vi** **Q** × **S** **vii** **Q** × **T** **viii** **R** × **S**
ix **R** × **T** **x** **S** × **T**

2 As always in algebra, if there is no operation sign between two terms, multiplication is to be assumed. **AK** means **A** × **K**, i.e. matrix **A** × matrix **K**.
Find the product of the matrices in each case below:

a $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

b $\begin{pmatrix} -1 & 2 & -1 \\ 0 & 0 & 3 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

c $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 5 \end{pmatrix}$

e $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}$

f $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 4 & 3 & 0 \\ 0 & -1 & 5 \end{pmatrix}$

g $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & 1 \end{pmatrix}$

h $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 4 & 2 \\ 6 & 0 \end{pmatrix}$

i $\begin{pmatrix} 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

j $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\begin{pmatrix} -2 & 3 & -1 \end{pmatrix}$

continued

continued

k $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

l $\begin{pmatrix} 7 & -2 & 5 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix}$

m $\begin{pmatrix} 2 & 0 \\ -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

n $\begin{pmatrix} 4 & 5 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$

o $\begin{pmatrix} -2 & -1 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$

p $\begin{pmatrix} 5 & 0 & -1 \\ 1 & 4 & 2 \\ -3 & -2 & 0 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 3 & -2 \\ 0 & 1 \end{pmatrix}$

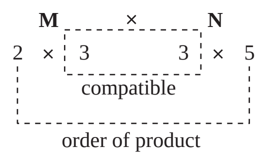
3 If $\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix}$, find: **a** $\mathbf{A}^2$, **b** $\mathbf{B}^3$.

23.6 COMPATIBILITY

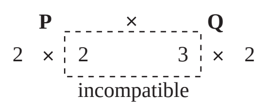
By now it has probably become clear to you that multiplication of two matrices is only possible when the number of columns in the first matrix equals the number of rows in the second matrix. i.e. the product $\underset{m\times n}{\mathbf{A}} \times \underset{p\times q}{\mathbf{B}} = \mathbf{C}$ exists only when $n = p$. When, and only when, this is so, the matrices are said to be 'compatible' for this multiplication. If $n \neq p$, the matrices cannot be multiplied and are said to be 'incompatible' for this operation.

Examples

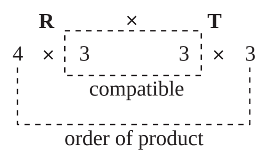
1 The product $\mathbf{M} \times \mathbf{N}$ does exist and has order 2×5.



2 The product $\mathbf{P} \times \mathbf{Q}$ does not exist.



3 The product $\mathbf{R} \times \mathbf{T}$ does exist and has order 4×3.



It is quite common for a product **A** × **B** to exist but for the product **B** × **A** *not* to exist. For example:

$\underset{2\times 3}{\mathbf{V}} \times \underset{3\times 3}{\mathbf{W}}$ exists (and has order 2 × 3), but

$\underset{3\times 3}{\mathbf{W}} \times \underset{2\times 3}{\mathbf{V}}$ does *not* exist.

It is easy to show that **A** × **B** and **B** × **A** both exist only for $\underset{m\times n}{\mathbf{A}} \times \underset{n\times m}{\mathbf{B}}$:

e.g. $\underset{2\times 3}{\mathbf{A}} \times \underset{3\times 2}{\mathbf{B}}$ and $\underset{3\times 2}{\mathbf{B}} \times \underset{2\times 3}{\mathbf{A}}$ both exist.

SUMMARY

Facts about the product of two matrices:

- **A** × **B** may not *exist.*
- Even if **A** × **B** does exist, it is possible that **B** × **A** does *not* exist.
- Even when both products exist, in general, **A** × **B** ≠ **B** × **A.**
- It is possible that, **A** × **B** = **0** even though neither **A** nor **B** is the zero matrix **O**.

Example

$$\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

You can verify this after studying the next section.

Exercises 23.6

1. Using the set of matrices in Exercises 23.3, state whether the given product exists in each case (answering 'yes' or 'no') and, if it *does* exist, state its order.

a **AB** **b** **BA** **c** **AC** **d** **DA** **e** **FE** **f** **EF**
g **CF** **h** **AE** **i** **BD** **j** **CD** **k** **DC** **l** **CE**

23.7 THE IDENTITY MATRIX, I

Exercises 23.7

1 Write down the product matrix:

a $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

continued

continued

2 Write down the product matrix:

a $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

The *principal diagonal* of a square matrix is the diagonal that runs from the top left-hand corner to the bottom right-hand corner. The square matrix, which has the number 1 for each element on the principal diagonal and all other elements zero, plays a very special role in the theory of matrices.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the *identity matrix* of order 2, and is specified as $\mathbf{I}_2$.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called the *identity matrix* of order 3, and is specified as $\mathbf{I}_3$.

For any square matrix $\mathbf{A}_n$ (i.e. of order $n \times n$), $\mathbf{A}_n \times \mathbf{I}_n = \mathbf{I}_n \times \mathbf{A}_n$.

$\mathbf{I}_n$ plays the same role in matrix theory as unity does in arithmetic (e.g. $7 \times 1 = 1 \times 7 = 7$), and so it is called the **unit matrix,** $\mathbf{U}_n$ or the **identity matrix,** $\mathbf{I}_n$. We use the latter name and symbol in this book.

We use capital letters to identify matrices, but the capital letter I is reserved for the *identity matrix* and the capital letter O is reserved for the *zero matrix.*

Note: For a matrix $\underset{n\times m}{\mathbf{A}}$ which is *not* square, $\underset{n\times m}{\mathbf{A}} \times \mathbf{I}_m = \underset{n\times m}{\mathbf{A}}$ (but $\mathbf{I}_m \times \underset{n\times m}{\mathbf{A}}$ does not exist) and $\mathbf{I}_n \times \underset{n\times m}{\mathbf{A}} = \underset{n\times m}{\mathbf{A}}$ (but $\underset{n\times m}{\mathbf{A}} \times \mathbf{I}_n$ does not exist).

Remember: I stands for the **I**dentity matrix, not the numeral 1.

Therefore, non-square matrices do not have an identity matrix.

Exercises 23.7 *continued*

3 For the following, write down the product matrix if it exists. If it does not exist, write 'incompatible'.

a $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ d $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4 a If $\begin{pmatrix} 17 & -13 & 19 \\ 34 & 28 & -15 \end{pmatrix} \times \mathbf{A} = \begin{pmatrix} 17 & -13 & 19 \\ 34 & 28 & -15 \end{pmatrix}$, write down the matrix **A**.

Remember: an identity matrix is always *square*.

b If $\mathbf{B} \times \begin{pmatrix} 17 & -13 & 19 \\ 34 & 28 & -15 \end{pmatrix} = \begin{pmatrix} 17 & -13 & 19 \\ 34 & 28 & -15 \end{pmatrix}$, write down the matrix **B**.

c Does $\begin{pmatrix} 17 & -13 & 19 \\ 34 & 28 & -15 \end{pmatrix}$ have an identity matrix? State the reason for your answer.

5 For each of the following matrices, state whether an identity matrix exists (answering 'yes' or 'no') and, if it *does* exist, write it down.

a $\begin{pmatrix} -3 & 2 \\ 7 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & 8 \\ 7 & 3 \\ 1 & 0 \end{pmatrix}$

c $\begin{pmatrix} 3 & -4 & 2 \\ 6 & 1 & -5 \end{pmatrix}$ **d** $\begin{pmatrix} 3 & -2 & 5 \\ 4 & 1 & 6 \\ -7 & 9 & 8 \end{pmatrix}$

6 If $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, show that: **a** $\mathbf{M}^2 = -\mathbf{I}$, **b** $\mathbf{M}^3 = -\mathbf{M}$.

23.8 THE INVERSE MATRIX, A^{-1}

The numbers 13 and $\frac{1}{13}$ are said to be multiplicative *inverses* of one another because in multiplication one *undoes* what the other *does*.
For example, $957 \times 13 \times \frac{1}{13} = 957$, $\sqrt{7} \times \frac{1}{13} \times 13 = \sqrt{7}$.
This occurs because $13 \times \frac{1}{13} = 1$ and $\frac{1}{13} \times 13 = 1$.
Now we will see how this applies to matrices.

Exercises 23.8

1 If $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}$, write down: **a** the matrix **AB**, **b** the matrix **BA.**

Since both products in the exercise above are the identity matrix, you can probably guess that **A** and **B** are said to be *inverses* of each other.

The inverse of matrix **M** is written as $\mathbf{M}^{-1}$, so you have proved for the above matrices **A** and **B** that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, that is, that $\mathbf{B} = \mathbf{A}^{-1}$ and $\mathbf{A} = \mathbf{B}^{-1}$.

Definition: Two matrices, **A** and **B**, are said to be inverses of one another (i.e. $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{B} = \mathbf{A}^{-1}$) if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

Exercises 23.8 *continued*

2 Given that $\mathbf{C} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$:

- **a** write down the matrix **CD**
- **b** write down the matrix **DC**
- **c** what have we proved about the matrices **C** and **D?**

3 Given that matrix $\underset{p\times q}{\mathbf{A}}$ has an inverse $\underset{r\times s}{\mathbf{B}}$:

- **a** what is the order of matrix **AB?**
- **b** what is the order of matrix **BA?**
- **c** Since these matrices are inverses of each other, by definition $\mathbf{AB} = \mathbf{BA} = \underset{n\times n}{\mathbf{I}}$. Therefore, the orders of **AB** and **BA** are both $n \times n$. What can you deduce about the values of p, q, r, s and n?
- **d** Hence, if matrices **A** and **B** are inverses of each other, what can you deduce about the *shapes* of the matrices **A** and **B?**

Most square matrices have an inverse, but not *all* of them. (Actually, it can be shown that all square matrices have an inverse **except those for which the determinant** $|\mathbf{A}| = 0$.)

A matrix that has an inverse is said to be **invertible.**

SUMMARY

- If $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A} = \mathbf{I}$, then **A** and **B** are inverses of each other (i.e. $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{B} = \mathbf{A}^{-1}$).
- A non-square matrix cannot have an inverse.
- Most, but not all, square matrices have an inverse (i.e. they are *invertible*).

Note: You are not required to be able to *find* the inverse $\mathbf{A}^{-1}$ of a given matrix **A**, but you should be able to determine whether or not two given matrices **A** and **B** are inverses of each other by testing whether $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

Exercises 23.8 *continued*

4 If $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$

$\mathbf{D} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ $\mathbf{E} = \begin{pmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$:

a write down the products:

i **AB** ii **AC** iii **AD** iv **AE**

v **BC** vi **BD** vii **BE**

b hence, write down: (i) matrix $\mathbf{A}^{-1}$ (ii) matrix $\mathbf{B}^{-1}$

5 Given that $\mathbf{J} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{pmatrix}$ and $\mathbf{K} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{pmatrix}$:

a write down the matrix **JK**

b write down the matrix $\mathbf{J}^{-1}$

6 Given that $\mathbf{P} = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 3 & -1 \\ 3 & 2 & 0 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & -4 & 4 \\ -3 & 6 & -1 \\ -7 & 4 & 1 \end{pmatrix}$:

a write down the matrix **PQ**

b write down the matrix $\mathbf{Q}^{-1}$

23.9 THE ALGEBRA OF MATRICES

As a result of the definitions of the operations of matrices, the laws for the algebra of matrices are mostly the same as the laws for the algebra of real numbers.

Name of law	Real numbers	Matrices
Commutative law for addition	$x + y = y + x$	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
Associative law for addition	$(x + y) + z = x + (y + z)$	$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
Identity law for addition	$x + 0 = 0 + x = x$ (0 is the identity element for addition)	$\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ (**O** is the identity matrix for addition, the *zero* matrix of **A**)
Identity law for multiplication	$x \times 1 = 1 \times x = x$ (1 is the identity element for multiplication)	$\mathbf{A} \times \mathbf{I} = \mathbf{I} \times \mathbf{A} = \mathbf{A}$ (**I** is the identity matrix for multiplication for matrix **A**)
Law of multiplicative inverse	$x \times x^{-1} = x^{-1} \times x = 1$ (x^{-1} is the multiplicative inverse of x)	$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$ ($\mathbf{A}^{-1}$ is the multiplicative inverse of **A**)
Distributive law for multiplication	$C(x + y) = Cx + Cy$ $(x + y)C = xC + yC$ $x(y + z) = xy + xz$ $(y + z)x = yx + zx$	$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$ $(\mathbf{A} + \mathbf{B})k = \mathbf{A}k + \mathbf{B}k$ $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$

Remember:

- **O, the zero matrix for matrix A**, has been defined as the matrix with the same order as **A** but having all its elements zeros.
- **I, the identity matrix for matrix A**, has been defined for square matrices only, being the matrix having the same order as **A**, with all the elements on the principal diagonal being 1 and all the other elements being zero.
- **A^{-1}, the inverse of matrix A**, has been defined for *square* matrices only, being the matrix such that $AA^{-1} = A^{-1}A = I$. The only matrices that have an inverse are *square matrices whose determinant* $\neq 0$.

Hence, most algebraic operations with matrices are already quite familiar to us.

Examples

1 If $3A + 4B = 5C$

then $3A = 5C - 4B$

$\therefore\ A = \frac{1}{3}(5C - 4B)$

2 $(A + B)(C + D) = AC + AD + BC + BD$

3 $(A + B)(A + I) = A^2 + AI + BA + BI$

$= A^2 + A + BA + B$

4 $A(A^{-1} + I) = AA^{-1} + AI$

$= I + A$, or $A + I$

5 $A(A^{-1}B) = (AA^{-1})B = IB = B$

6 $AB + A = AB + AI$

$= A(B + I)$, *not* $A(B + 1)$ because we cannot add a real number to a matrix

7 $A^{-1}(A + I) = A^{-1}A + A^{-1}I$

$= I + A^{-1}$, or $A^{-1} + I$

However, there is a difference between the algebra for matrices and the algebra for real numbers when we are *multiplying* because, as we have already noted, in general:

$A \times B \neq B \times A.$

(Don't forget that this statement does not mean that they can *never* be equal but that we cannot *assume* they *are* equal, because *usually* they are not equal.)

Hence, care must be taken to maintain the *correct order* of matrices when multiplying.

For example, $A(B + C) \neq (B + C)A$ and $ABA^{-1} \neq A^{-1}AB$ (which would equal B).

The only occasion when multiplying matrices is commutative is when they are inverses of one another ($AA^{-1} = A^{-1}A$ [$=I$]) or when one of them is the identity matrix ($AI = IA$ [$= A$]).

Example

Make **K** the subject of the equation $AK = B$.

We proceed thus: $AK = B$

$\therefore\ A^{-1}(AK) = A^{-1}B$ (*not* BA^{-1})

$\therefore\ (A^{-1}A)K = A^{-1}B$

$\therefore\ K = A^{-1}B$

Note: We cannot add a matrix and a real number (e.g. **A** + 2 makes no sense). But we can always replace **A** by **AI** or by **IA**.

Example

Solve the matrix equation $\mathbf{AF} + 2\mathbf{F} = \mathbf{B}$, for $\mathbf{F}$.

$$\mathbf{AF} + 2\mathbf{F} = \mathbf{B}$$
$$\therefore \mathbf{AF} + 2\mathbf{IF} = \mathbf{B}$$
$$(\mathbf{A} + 2\mathbf{I})\mathbf{F} = \mathbf{B}$$
$$\therefore (\mathbf{A} + 2\mathbf{I})^{-1}(\mathbf{A} + 2\mathbf{I})\mathbf{F} = (\mathbf{A} + 2\mathbf{I})^{-1}\mathbf{B}$$
$$\therefore \mathbf{F} = (\mathbf{A} + 2\mathbf{I})^{-1}\mathbf{B}$$

Exercises 23.9

1 Simplify, removing all brackets:
   a $\mathbf{A}(\mathbf{A}^{-1}\mathbf{B})$
   b $\mathbf{A}(\mathbf{A}^{-1}\mathbf{I})$
   c $\mathbf{A}(\mathbf{B}\mathbf{A}^{-1}) + \mathbf{A}(\mathbf{A}^{-1}\mathbf{B})$
   d $\mathbf{AB}(\mathbf{A}^{-1}\mathbf{B} + \mathbf{B}^{-1}\mathbf{A})$
   e $\mathbf{I}^2$
   f $\mathbf{I}^2 + \mathbf{I}^3$
   g $(\mathbf{A} + \mathbf{I})^2$
   h $(\mathbf{A} + \mathbf{I})(\mathbf{B} + \mathbf{A})$
   i $\mathbf{A}(\mathbf{I} - \mathbf{A}^{-1}) + \mathbf{I}^2$
   j $\mathbf{A}(\mathbf{A}^{-1} + \mathbf{I}) - \mathbf{B}^{-1}\mathbf{B} - \mathbf{IA}$

2 Solve the following matrix equation for **X**:
   a $2\mathbf{X} - \mathbf{A} = \mathbf{B}$
   b $\mathbf{C} - 3\mathbf{X} = \mathbf{D}$
   c $3(\mathbf{A} - 2\mathbf{X}) = 2(3\mathbf{A} - \mathbf{X})$
   d $\dfrac{\mathbf{X} + \mathbf{A}}{3} = \dfrac{\mathbf{X} - 3\mathbf{B}}{2}$

3 Solve the following matrix equations for **X**:
   a $\mathbf{AX} = \mathbf{B}$
   b $\mathbf{XA} = \mathbf{B}$
   c $2\mathbf{A} + \mathbf{AX} = \mathbf{B}$
   d $\mathbf{AX} + \mathbf{I} = \mathbf{B}$
   e $\mathbf{X}^{-1} = \mathbf{A}$
   f $3\mathbf{X} - \mathbf{AX} = \mathbf{B}$
   g $\mathbf{AX} + \mathbf{X} = \mathbf{B}$
   h $\mathbf{AX} + \mathbf{B} = \mathbf{C} - \mathbf{X}$
   i $\mathbf{X} + \mathbf{XA} = \mathbf{B}$
   j $2\mathbf{X} + \mathbf{B} = \mathbf{C} - 2\mathbf{XA}$

4 If $\mathbf{AB} = \mathbf{AC}$:
   a does it follow that $\mathbf{B} = \mathbf{C}$ ('yes' or 'no')?
   b and $\mathbf{B} \neq \mathbf{C}$, what *is* **B** equal to?

5 If $\mathbf{AB} = \mathbf{CA}$:
   a does it follow that $\mathbf{B} = \mathbf{C}$ ('yes' or 'no')?
   b and $\mathbf{B} \neq \mathbf{C}$, what *is* **B** equal to?

Note:

- The matrix, $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, the associative law, **but we must not change the order of the matrices when they are being multiplied.**
- The matrices **ABC, ACB, BCA, BAC, CAB** and **CBA** are probably all different matrices.
- However, consideration will show you that if *k* is a constant (i.e. a real number), then $k \times (\mathbf{A} \times \mathbf{B}) = (k \times \mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (k \times \mathbf{B})$. Which matrix we multiply by *k*, either $\mathbf{A} \times \mathbf{B}$ or **A** or **B,** makes no difference to the final result. Although we must not change the positions of the *matrices*, we may change the position of a constant, *k*.
- Note also that $k \times \mathbf{A} = k \times \mathbf{AI} = \mathbf{A} \times k\mathbf{I}$.

Exercises 23.9 *continued*

6 Solve the following matrix equations for **X**:

a $3\mathbf{X} + 2\mathbf{XA} = \mathbf{C} - \mathbf{B}$ **b** $\mathbf{A} + 2\mathbf{X} = \mathbf{B} + 3\mathbf{XC}$

23.10 EXPRESSING SIMULTANEOUS EQUATIONS IN MATRIX FORM

Exercises 23.10

1 Find the following products and state their order:

a $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 5 & -2 \\ 1 & 4 & 3 \\ 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2 Express as the product of two matrices:

a $\begin{pmatrix} 3x + 4y \\ 2x - 5y \end{pmatrix}$ **b** $\begin{pmatrix} 2x - 3y + 4z \\ x + 2y - 5z \\ 3x - y + 2z \end{pmatrix}$ **c** $\begin{pmatrix} 3a - 5b + 2c \\ 2a + 4b \\ 5a - 7c \end{pmatrix}$

3 Solve for *x* and *y*:

a $\begin{pmatrix} 2x \\ 3y \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$ **b** $\begin{pmatrix} 13 & 1 \\ 12 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 29 \\ 27 \end{pmatrix}$

4 Express as a system of simultaneous equations:

a $\begin{pmatrix} 2x + 3y \\ 5x - 2y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 5 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

c $\begin{pmatrix} 3x - 2y + 4z \\ 2x + 3y - 5z \\ x - 4y + 2z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

5 Express each of the following systems of simultaneous equations as a single matrix equation:

a $\begin{cases} 5a + 7b - 3c = 17 \\ 7a - 2b - 8c = 13 \\ 3a + 5b + 5c = 19 \end{cases}$ **b** $\begin{cases} 4x - 5y + 6z = 9 \\ 7x + 3y = 5 \\ 3x - 8z = 7 \end{cases}$

23.11 SOLVING SIMULTANEOUS LINEAR EQUATIONS USING MATRICES

If a set of simultaneous equations is represented in matrix form

$\mathbf{C} \times \mathbf{U} = \mathbf{K}$

where:

C is the matrix formed by the coefficients

U is the matrix formed by the unknowns

K is the matrix formed by the constants

then the solution to the matrix of unknown constants **U** can be solved by multiplying both sides of the matrix equation by the inverse of **C**, the matrix of coefficients.

$\mathbf{C}^{-1} \times \mathbf{C} \times \mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$

We know that:

$\mathbf{C}^{-1} \times \mathbf{C} = \mathbf{I}$ (the identity matrix)

Therefore,

$\mathbf{I} \times \mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$

We also know that:

$\mathbf{I} \times \mathbf{U} = \mathbf{U}$

Therefore the solution to **U**, the matrix of unknown constants, becomes:

$\mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$

This is a simple but powerful equation. In order to find the solution to a matrix of unknowns we only need to determine the inverse of the matrix of coefficients.

Solving two equations with two unknowns

For the general equation

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

represented in matrix equation form by

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{K} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\mathbf{C} \times \mathbf{U} = \mathbf{K}$ or $\mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$

Example

Solve the simultaneous equations

$$\begin{cases} 5x - 2y = 6 \\ 3x - y = 5 \end{cases}$$

given that the inverse of $\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$ is $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$.

continued

continued

Steps

1 Express the equations as a single matrix equation:

$$\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

2 Multiply both sides by the inverse matrix:

$$\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

3 Multiply out the matrices on each side—we *know* the product on the left-hand side because

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\therefore x = 4, y = 7$$

Warning:

The error most often made is in step 3. Remember that $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ is not the same as $\begin{pmatrix} 6 \\ 5 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$ because for matrices $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$.

Finding the inverse of a 2 × 2 matrix

In order to find the solution to the matrix of unknowns **U** we need to find the inverse of the matrix of coefficients $\mathbf{C}^{-1}$.

To solve for $\mathbf{C}^{-1}$ we manipulate the matrix of coefficients **C** using the following steps:

1 Interchange the elements on the principal diagonal.
2 Reverse the signs of the elements on the secondary diagonal.
3 Divide by the determinant of the original matrix.

For $\mathbf{C} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $\mathbf{C}^{-1} = \dfrac{1}{|\mathbf{C}|}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$

Example

Solve the following set of linear equations:

$-4x + 3y = 2$

$-2x + y = 4$

Representing in matrix form:

$$\begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$\mathbf{C} \times \mathbf{U} = \mathbf{K}$

Manipulate the matrix of the coefficients **C**:

$$\mathbf{C} = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix}$$

Interchange the elements on the principal diagonal and reverse the signs of the elements on the secondary diagonal.

$$\mathbf{C}^{-1} = \frac{1}{|\mathbf{C}|}\begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$$

Calculate the determinant of the matrix of coefficients.

$$|\mathbf{C}| = \begin{vmatrix} -4 & 3 \\ -2 & 1 \end{vmatrix} = (-4) - (-6) = 2$$

Divide by the determinant of the matrix of coefficients.

$$\mathbf{C}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$$

Now that we have determined $\mathbf{C}^{-1}$ we can continue and solve for the matrix of unknowns **U**.
Multiply both sides of the matrix equation by $\mathbf{C}^{-1}$:

$$\mathbf{C}^{-1} \times \mathbf{C} \times \mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$$

$$\mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times (2 - 12) \\ \frac{1}{2} \times (4 - 16) \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \end{pmatrix}$$

$$\therefore x = -5, y = -6$$

Note: If $|\mathbf{A}| = 0$, the matrix **A** has no inverse (because division by zero is not defined). Therefore, the matrix is not invertible.

Solving three equations with three unknowns

The general equations are:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Representing in matrix form:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_e \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{K} = \begin{pmatrix} d_1 \\ d_2 \\ d_e \end{pmatrix}$$

$\mathbf{C} \times \mathbf{U} = \mathbf{K}$, where **U** is the matrix of unknowns, given by $\mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$

Example

If $\mathbf{M} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{pmatrix}$ and $\mathbf{N} = \frac{1}{2}\begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix}$:

a find the matrix **MN**

b write down the matrix $\mathbf{M}^{-1}$

c express the given system linear equations as a single matrix equation and use the result of **b** above to solve these simultaneous equations:

$$\begin{cases} x + 2y - z = -1 \\ 3x + 5y - z = 2 \\ -2x - y - 2z = -9 \end{cases}$$

Solutions

a $\mathbf{MN} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\mathbf{M}^{-1} = \frac{1}{2}\begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix}$

c $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -9 \end{pmatrix}$

$$\mathbf{M} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -9 \end{pmatrix}$$

$$\mathbf{M}^{-1}\mathbf{M} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -9 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 6 \\ -2 \\ -4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$\therefore x = 3, y = -1, z = 2$

Finding the inverse of a 3 × 3 matrix

As with a 2 × 2 matrix, the solution to **U**, the matrix of unknowns, first requires us to be able to determine $\mathbf{C}^{-1}$. Again, to find $\mathbf{C}^{-1}$ we manipulate the matrix **C,** but in the case of a 3 × 3 matrix the manipulation is more complex. The solution however is highly methodical and lends itself to the use of spreadsheets or computer programs. The inverse of a 3 × 3 matrix of coefficients, C^{-1}, = the matrix of minors ÷ the determinant of the matrix of coefficients.

$$\mathbf{C}^{-1} = \frac{1}{|\mathbf{C}|} \times \begin{pmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix}$$

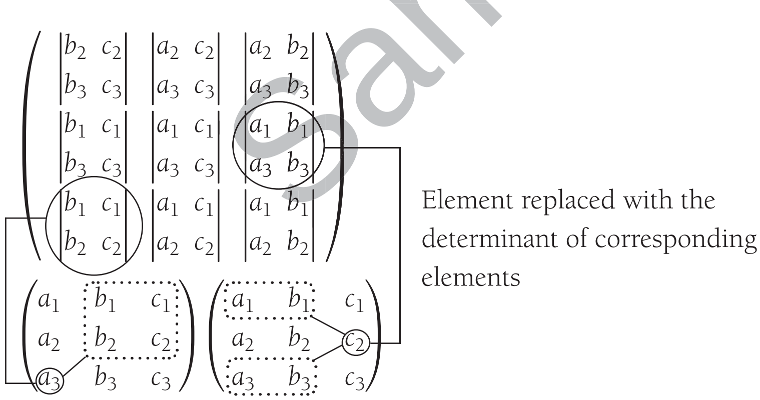
The matrix of minors

where $|\mathbf{C}|$ is the determinant of the matrix of coefficients

$$\text{and } |\mathbf{C}| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

For computer applications such as spreadsheets and programs the above can be used directly and the calculations are straightforward. For hand calculations however it becomes difficult to remember the position of the determinants in the matrix and the order of the elements in the determinants, so a stepwise approach for hand calculations is the best approach as follows.

1 Calculate the matrix of minors. To develop the matrix of minors, each element in the matrix is replaced with the determinant of corresponding matrix elements. The corresponding elements are the elements that are not in the same row or the same column as the element being replaced.



Element a_3 is replaced with the determinant of elements b_1, c_1, b_2, c_2.
Element c_2 is replaced with the determinant of elements a_1, b_1, a_3, b_3.

2 The sign of each element in the matrix of minors is modified according to the following matrix of coefficients.

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \end{array}$$

This means that the element in R_1C_1 remains unchanged, i.e. it is multiplied by $+1$. The element in R_1C_2 changes sign, i.e. it is multiplied by -1.

$$\begin{pmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix}$$

3 The matrix is transposed. This means that the elements in the columns of the matrix become the elements in the rows and vice versa.

$$\begin{pmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix}$$

4 The fourth and final step is to divide this manipulated matrix by the determinant of the original matrix. This provides us with the inverse of the matrix.

$$\mathbf{C}^{-1} = \frac{1}{|\mathbf{C}|}\begin{pmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix}$$

where:

$$|\mathbf{C}| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example

Solving the following set of linear equations:

$$\begin{cases} x - 2y + 3z = 4 \\ 3x + y - 2z = -7 \\ 4x - 4y + 3z = -3 \end{cases}$$

Representing in matrix form

$$\begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 4 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix}$$

$\mathbf{C} \times \mathbf{U} = \mathbf{K}$

$$\mathbf{C} = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 4 & -4 & 3 \end{pmatrix}$$

Step 1 Solve for $\mathbf{C}^{-1}$.

Find the matrix of minors.

$$\begin{pmatrix} \begin{vmatrix} 1 & -2 \\ -4 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 4 & -4 \end{vmatrix} \\ \begin{vmatrix} -2 & 3 \\ -4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 4 & -4 \end{vmatrix} \\ \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}$$

$$\begin{pmatrix} (1 \times 3) - (-2 \times -4) & (3 \times 3) - (-2 \times 4) & (3 \times -4) - (1 \times 4) \\ (-2 \times 3) - (3 \times -4) & (1 \times 3) - (3 \times 4) & (1 \times -4) - (-2 \times 4) \\ (-2 \times -2) - (3 \times 1) & (1 \times -2) - (3 \times 3) & (1 \times 1) - (-2 \times 3) \end{pmatrix}$$

$$\begin{pmatrix} -5 & 17 & -16 \\ 6 & -9 & 4 \\ 1 & -11 & 7 \end{pmatrix}$$

Step 2 Modify according to the matrix of coefficients.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \begin{pmatrix} -5 & -17 & -16 \\ -6 & -9 & -4 \\ 1 & 11 & 7 \end{pmatrix}$$

Step 3 Transpose the matrix.

$$\begin{pmatrix} -5 & -6 & 1 \\ -17 & -9 & 11 \\ -16 & -4 & 7 \end{pmatrix}$$

Step 4 Divide this matrix by the determinant $|\mathbf{C}|$ of the original matrix to obtain $\mathbf{C}^{-1}$.

$$|\mathbf{C}| = 1 \begin{vmatrix} 1 & -2 \\ -4 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 3 & -2 \\ 4 & 3 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 \\ 4 & -4 \end{vmatrix}$$

$$= 1 \times (1 \times 3) - (-2 \times -4) - (-2) \times (3 \times 3) - (-2 \times 4) + 3 \times (3 \times -4) - (1 \times 4)$$

$$= -19$$

$$\mathbf{C}^{-1} = \frac{1}{-19} \begin{pmatrix} -5 & -6 & 1 \\ -17 & -9 & 11 \\ -16 & -4 & 7 \end{pmatrix}$$

Step 5 Multiply both sides of the matrix equation by the inverse matrix.

$$\mathbf{C}^{-1} \times \mathbf{C} \times \mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$$

$$\mathbf{U} = \mathbf{C}^{-1} \times \mathbf{K}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{C}^{-1} \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-19} \begin{pmatrix} -5 & -6 & 1 \\ -17 & -9 & 11 \\ -16 & -4 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix}$$

continued

continued

$$= \tfrac{1}{-19}\begin{pmatrix}(-5\times 4) + (-6\times -7) + (1\times -3)\\(-17\times 4) + (-9\times -7) + (11\times -3)\\(-16\times 4) + (-4\times -7) + (7\times -3)\end{pmatrix}$$

$$= \tfrac{1}{-19}\begin{pmatrix}19\\-38\\-57\end{pmatrix}$$

$$= \begin{pmatrix}-1\\2\\3\end{pmatrix}$$

$x = -1, y = 2, z = 3$

Exercises 23.11

1 **a** Express the simultaneous equations

$2x - 3y = 9$

$5x - 7y = 22$

as a single matrix equation.

b Solve the above equations using matrices, given that the inverse of the matrix $\begin{pmatrix}2 & -3\\5 & -7\end{pmatrix}$ is $\begin{pmatrix}-7 & 3\\-5 & 2\end{pmatrix}$.

2 You are given that if $\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} = \begin{pmatrix}d & -b\\-c & a\end{pmatrix}$.

a If $\mathbf{P} = \begin{pmatrix}3 & 2\\5 & 4\end{pmatrix}$:

i evaluate $|\mathbf{P}|$

ii write down the matrix $\mathbf{P}^{-1}$

b Use the result for $\mathbf{P}^{-1}$ above to solve the system of equations

$$\begin{cases}3x + 2y = 4\\5x + 4y = 10\end{cases}$$

3 If $\mathbf{M} = \begin{pmatrix}4 & 3\\1 & -2\end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix}2 & 3\\1 & -4\end{pmatrix}$:

a find the product **MN**

b write down the matrix $\mathbf{M}^{-1}$

c use the result of **a** above to solve the simultaneous equations below, showing each step of the working:

$$\begin{cases}4x + 3y = 7\\x - 2y = 10\end{cases}$$

4 For the following systems of simultaneous equations:

i express the equations in matrix form

ii find the determinant

iii find the inverse matrix

iv solve the equations

a $\begin{aligned} a - 3b &= -1 \\ 4a - 11b &= -2 \end{aligned}$ **b** $\begin{aligned} a - b &= 6 \\ 2a + 3b &= 2 \end{aligned}$ **c** $\begin{aligned} 3x - y &= 6 \\ -2x + y &= 2 \end{aligned}$

d $\begin{aligned} 4p + 4q &= 8 \\ 5p - 3q &= 2 \end{aligned}$ **e** $\begin{aligned} x - 3y &= 4 \\ 4x + 8y &= 6 \end{aligned}$

5 The inverse of $\begin{pmatrix} 4 & -1 & 1 \\ 3 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 7 \\ -1 & 3 & -5 \end{pmatrix}$. Use this result to solve the system of equations $\begin{cases} 4a - b + c = 15 \\ 3a - 2b - c = 13 \\ a - b - c = 3 \end{cases}$

6 If $\mathbf{P} = \begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q} = \frac{1}{2}\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{pmatrix}$:

a write down the matrix **PQ**

b write the system of equations below as a single matrix equation

$$\begin{cases} 11x - 5y - 3z = -12 \\ -8x + 4y + 2z = 10 \\ -7x + 3y + z = 10 \end{cases}$$

c use the result of **a** above to solve the simultaneous equations in **b** using matrices.

7 If $\mathbf{P} = \begin{pmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$:

a write down the matrices (i) **PQ** (ii) **PR** (iii) **QR**

b which two of these matrices are inverses of each other?

c express the simultaneous equations $\begin{cases} a + 2b - 2c = 3 \\ 2a + 5b - 4c = 7 \\ 3a + 7b - 5c = 8 \end{cases}$ as a single matrix equation

d solve the equations in **c** using matrices.

8 Solve the following systems of simultaneous equations using matrices:

a $\begin{cases} 3x + 2y - 2z = -3 \\ 2x + 2y - z = 1 \\ -4x - 3y + 2z = 0 \end{cases}$ given that the inverse of $\begin{pmatrix} 3 & 2 & -2 \\ 2 & 2 & -1 \\ -4 & -3 & 2 \end{pmatrix}$ is $\begin{pmatrix} -1 & -2 & -2 \\ 0 & 2 & 1 \\ -2 & -1 & -2 \end{pmatrix}$

b $\begin{cases} 2a - 4b - c = 0 \\ 5a - 10b - 3c = 1 \\ 15a - 29b - 9c = 5 \end{cases}$ given that if $\mathbf{K} = \begin{pmatrix} 2 & -4 & -1 \\ 5 & -10 & -3 \\ 15 & -29 & -9 \end{pmatrix}$, then $\mathbf{K}^{-1} = \begin{pmatrix} 3 & -7 & 2 \\ 0 & -3 & 1 \\ 5 & -2 & 0 \end{pmatrix}$

c $\begin{cases} 2p + 3q + 3r = -2 \\ 3p + 5q + 5r = -4 \\ 5p + 3q + 4r = 0 \end{cases}$ given that $\begin{pmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -3 & 0 \\ 13 & -7 & -1 \\ -16 & 9 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

continued

continued

d $\begin{cases} 2x + 3y + 2z = 1 \\ 3x + 4y + 3z = 1 \\ 5x + y + 4z = -1 \end{cases}$ given that $\begin{pmatrix} 2 & 3 & 2 \\ 3 & 4 & 3 \\ 5 & 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 13 & -10 & 1 \\ 3 & -2 & 0 \\ -17 & 13 & -1 \end{pmatrix}$

e $\begin{cases} 23x + 5y - 35z = -2 \\ 13x + 3y - 20z = -1 \\ 19x + 4y - 29z = -2 \end{cases}$ given that $\begin{pmatrix} 7 & -5 & -5 \\ 3 & 2 & -5 \\ 5 & -3 & -4 \end{pmatrix}\begin{pmatrix} 23 & 5 & -35 \\ 13 & 3 & -20 \\ 19 & 4 & -29 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

f $\begin{cases} a + 2b + 3c = -2 \\ 2a + 3b + 4c = 0 \\ 3a + b - 2c = 17 \end{cases}$ given that $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} -10 & 7 & -1 \\ 16 & -11 & 2 \\ -7 & 5 & -1 \end{pmatrix} = \mathbf{I}$

g $\begin{cases} 2a + 2b = -6 \\ 3a + 2b + c = 0 \\ 7a + 5b + 2c = -1 \end{cases}$ given that if $\mathbf{M} = \begin{pmatrix} -1 & -6 & 3 \\ 1 & 4 & -2 \\ 1 & 11 & -5 \end{pmatrix}$, then $\mathbf{M}^{-1} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 1 \\ 7 & 5 & 2 \end{pmatrix}$

+++ **9** For the following systems of simultaneous equations:

i express the equations in matrix form

ii find the determinant

iii find the inverse matrix

iv solve the equations

a $\begin{cases} 2x - 2y - z = 1 \\ x - 5y - 3z = 2 \\ x + 6y + 4z = -3 \end{cases}$ **b** $\begin{cases} 2a + 3b + 2c = -4 \\ 2a + 2b + 2c = -2 \\ 5a + b + 4c = -6 \end{cases}$ **c** $\begin{cases} p + q + r = 4 \\ 2p - 2q + r = 5 \\ 2p + 2q + 4r = 6 \end{cases}$

d $\begin{cases} 2x + 2y - 2z = -2 \\ -2x + 2y - z = 1 \\ -4x - 3y + 3z = 0 \end{cases}$ **e** $\begin{cases} 4a + 2b - 4c = -4 \\ 4a - 5b + 2c = -2 \\ 5a - 2b - c = -5 \end{cases}$

23.12 3 × 3 DETERMINANTS: DEFINITION AND EVALUATION

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is a square array of 'elements' having three rows and three columns. It is called a '3 × 3 determinant' or a 'determinant of third order'.

The value of a **second-order** determinant may be defined to provide a shorthand method of solving two simultaneous equations in two unknowns. The value of a **third-order** determinant is defined so that it provides a shorthand method of solving **three** simultaneous equations in three unknowns.

The value of the above determinant is defined as $aei + bfg + cdh - gec - hfa - idb$. There are several ways of obtaining this result without having the very difficult task of committing it to memory. We will use the method called the *Rule of Sarrus*. This method is the simplest but it *applies only to* *determinants of order* ***three.*** (Later, when you study determinants of higher orders, you will learn other methods and ones that are easier to program for a computer.)

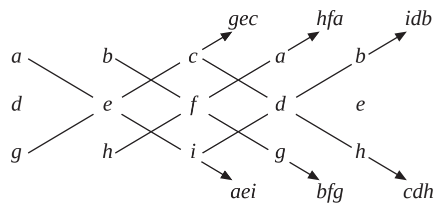
The Rule of Sarrus

1 Write down the determinant, repeating the first two columns.
2 Obtain the products on the diagonals as shown below.
3 Add the lower products and add the upper products.
4 Subtract the sum of the upper products from the sum of the lower products.

Follow the application of this rule as we apply it to the general determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

1 $\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$

2, 3 $(Sum = gec + hfa + idb)$



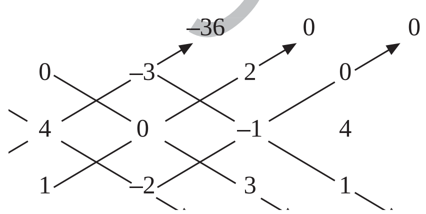
$(Sum = aei + bfg + cdh)$

4 Value: $(aei + bfg + cdh) - (gec + hfa + idb)$

Example

Evaluate: $\begin{vmatrix} 2 & 0 & -3 \\ -1 & 4 & 0 \\ 3 & 1 & -2 \end{vmatrix}$

Method: $(Sum = -36)$



$(Sum = -13)$

Value: $= (-13) - (-36)$
$= 23$

Hint: When copying down a determinant be very careful to include any negative signs. *Check* your copy before working on it. If you copied it row by row, check it column by column. It is very annoying to work on data that is later discovered to have been copied down incorrectly.

Exercises 23.12

1 Evaluate: (**Note:** All the elements are exact numbers.)

a $\begin{vmatrix} 2 & 1 & -2 \\ 3 & 0 & -1 \\ 5 & 0 & 1 \end{vmatrix}$ **b** $\begin{vmatrix} 4 & 3 & 5 \\ 0 & 5 & 0 \\ -4 & 4 & -5 \end{vmatrix}$ **c** $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{vmatrix}$ **d** $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & -8 \\ -5 & -2 & 1 \end{vmatrix}$

e $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$ **f** $\begin{vmatrix} 0 & t & n \\ -t & 0 & x \\ -n & -x & 0 \end{vmatrix}$ **g** $\begin{vmatrix} 42 & 28 & 62 \\ 36 & 29 & 91 \\ 37 & 30 & 47 \end{vmatrix}$ **h** $\begin{vmatrix} 20 & -24 & 28 \\ -31 & 47 & -64 \\ 83 & -51 & -86 \end{vmatrix}$

2 In each case below, evaluate the pronumeral:

a $\begin{vmatrix} x & 2x & 3x \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 9$ **b** $\begin{vmatrix} 1 & x & 2 \\ 2 & 0 & x \\ 3 & 1 & 1 \end{vmatrix} = 10$ **c** $\begin{vmatrix} 2 & n & 1 \\ 1 & n & 2 \\ 1 & n & 1 \end{vmatrix} = 5$

d $\begin{vmatrix} t & t & 1 \\ 4 & 1 & 0 \\ -2 & t & t \end{vmatrix} = 3$ **e** $\begin{vmatrix} 1 & 0 & k \\ -1 & 2 & 2 \\ k & 2 & 0 \end{vmatrix} = \begin{vmatrix} -1 & k & 6 \\ 1 & 0 & -2 \\ -2 & 1 & k \end{vmatrix}$

23.13 SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS USING 3 × 3 DETERMINANTS

A system of three linear equations in three unknowns may be solved algebraically by the elimination method.

However, this method is usually very tedious and the equations are more easily solved using determinants.

The general equations are: $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$

If we solve these equations by elimination we obtain the solution:

$$x = \frac{\triangle x}{\triangle},\ y = \frac{\triangle y}{\triangle},\ z = \frac{\triangle z}{\triangle}$$

where: $\triangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, which is the determinant formed by using the coefficients on the left-hand side of the equation

$\triangle_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, which is the same determinant but with the coefficients of x replaced by the constants (i.e. the numbers on the right-hand side)

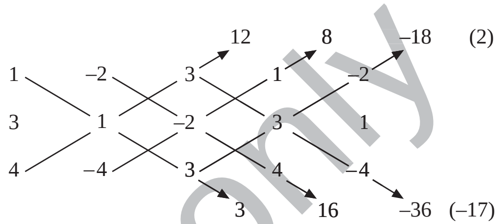
$$\triangle_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$, which is $\triangle$ again but with the coefficients of y replaced by the constants

$$\triangle_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$, which is $\triangle$ again but with the coefficients of z replaced by the constants.

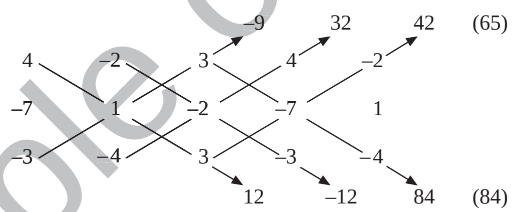
Example

Given: $$\begin{cases} x - 2y + 3z = 4 \\ 3x + y - 2z = -7 \\ 4x - 4y + 3z = -3 \end{cases}$$

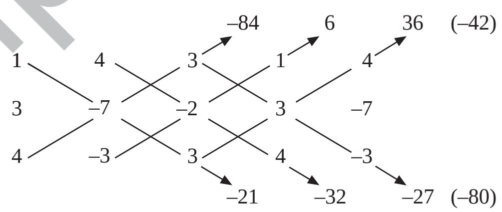
$$\triangle = \begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 4 & -4 & 3 \end{vmatrix} = (-17) - 2 = -19$$



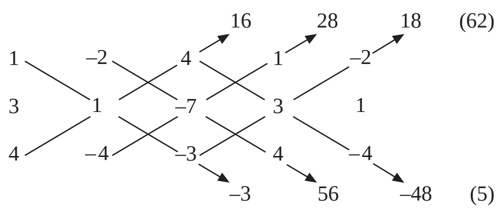
$$\triangle_x = \begin{vmatrix} 4 & -2 & 3 \\ -7 & 1 & -2 \\ -3 & -4 & 3 \end{vmatrix} = (84) - (65) = 19$$



$$\triangle_y = \begin{vmatrix} 1 & 4 & 3 \\ 3 & -7 & -2 \\ 4 & -3 & 3 \end{vmatrix} = (-80) - (-42) = -38$$



$$\triangle_z = \begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & -7 \\ 4 & -4 & -3 \end{vmatrix} = (5) - (62) = -57$$



Solution

$$x = \frac{\triangle_x}{\triangle} = \frac{19}{-19} = -1$$

$$y = \frac{\triangle_y}{\triangle} = \frac{-38}{-19} = 2$$

$$z = \frac{\triangle_z}{\triangle} = \frac{-57}{-19} = 3$$

Points to note

- After much practice, the value of a determinant can be found on a calculator, using the *Rule of Sarrus*, showing no intermediate results. The diagonal products can be summed using the M^+ and M^- keys. However, for the time being, you are advised to *write down* the product of each diagonal before adding it to the calculator memory, as in the examples given. This allows you to check your work more easily and also allows for credit to be given in an examination for knowledge of the *method* even if an error is made during the computation.
- When you have solved a set of simultaneous equations, check your result by substituting your values back into the original equations. You will be surprised how often careless errors are made during a long series of calculations.
- When using a calculator, always work with *all* the significant figures in the data. State your result to the *appropriate* number of significant figures, that is, the number required or justified.

Hints: **a** Before using the *Rule of Sarrus* to solve a set of simultaneous equations, make sure that:

- **i** all the equations have the pronumerals in the *same order*
- **ii** all the pronumerals are on the left-hand sides of the equations and all the constants are on the right-hand sides
- **iii** if a pronumeral is *absent* from an equation, write it in with a zero coefficient—for example, if there is no y in an equation, write it in as $0y$.

b When copying down an array to work on, be careful to place the elements in a neat rectangle so that the diagonal elements are approximately in straight lines. An untidy array leads to errors in computation.

c Before working on your arrays, check that you have made no errors, especially by omitting any negative signs. Discover any errors *before* you start to multiply.

d When multiplying out diagonals, either mentally or with a calculator, ignore any negative signs present. Decide *after* the multiplication whether the product should be positive or negative.

Exercises 23.13

1 Solve for the pronumerals using determinants (do not use a calculator):

a $\begin{cases} 2x + y + 3z = 4 \\ 3x - y - 4z = 5 \\ 4x + 3y + 2z = 1 \end{cases}$

b $\begin{cases} 2p + 4q + 3r + 8 = 0 \\ p + 3q - 2r + 2 = 0 \\ 3p - 5q - 4r - 4 = 0 \end{cases}$

c $\begin{cases} 3x + 2y + 4z = -7 \\ 2y - 3z + 4x = 6 \\ 4z - 5x - 2y = -7 \end{cases}$

d $\begin{cases} 4n - 3p + 5t = -3 \\ 3n + 4p = -6 \\ 5p - 4t = -4 \end{cases}$

2 Solve for the pronumerals using a calculator:

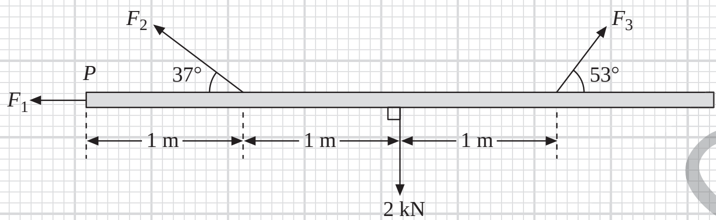
a $\begin{cases} 37x + 49y + 76z = 98 \\ 68y - 34x - 93z = 136 \\ 82z - 36y + 29x = -72 \end{cases}$

b $\begin{cases} 0.002x + 0.003y + 0.001z = 0.01 \\ 0.3x - 0.4y + 0.2z = 0.8 \\ 4000x + 5000y - 3000z = 4000 \end{cases}$

Hint: In **b**, multiply or divide both sides of each equation by a constant so as to obtain simpler numbers. State results correct to 3 significant figures.

c $\begin{cases} 2(3.7k - 1.4t) + 3(4.3t + 2.6w) + 8.7 = 0 \\ 4(2.3k + 1.7w) - 5.3w - 6.1 = 0 \\ 3(3.9k + 4.2t) = 0 \end{cases}$

3



The equations for equilibrium of this beam are:

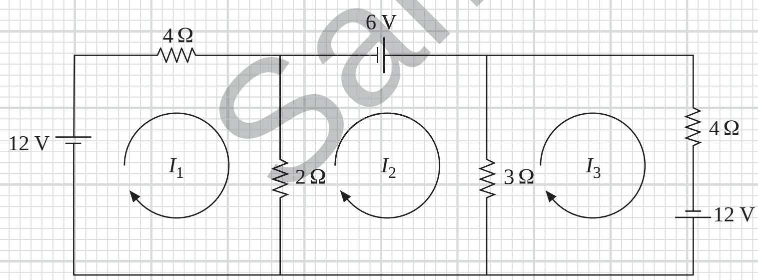
Vertical forces: $0.6F_2 - 2 + 0.8F_3 = 0$

Horizontal forces: $F_1 + 0.8F_2 - 0.6F_3 = 0$

Moments about point P: $0.6F_2 - 4 + 2.4F_3 = 0$

Using determinants, solve for F_1, F_2 and F_3, stating the results correct to 3 significant figures. Explain the meaning of any negative values.

4



Applying Kirchhoff's laws to this network:

$$4I_1 + 2(I_1 - I_2) = 12$$
$$2(I_2 - I_1) + 3(I_2 - I_3) = 6$$
$$3(I_3 - I_2) + 4I_3 = 12$$

Use determinants to solve for I_1, I_2 and I_3, stating their values correct to 3 significant figures.

continued

continued

5 A developer receives council permission to divide his land of area 44 ha into 100 blocks, the only areas allowed for a block being 2 ha, 0.5 ha and 0.2 ha. He prices the 2 ha blocks at $100 000 each, the 0.5 ha blocks at $40 000 each and the 0.2 ha blocks at $20 000 each. He sells all the blocks, the total gross income from the sales being $3 200 000. How many blocks of each size did he sell?

6 A firm has a stock of three different bronze alloys. Alloy A consists of 95% copper, 3% tin and 2% zinc. Alloy B consists of 90% copper, 9% tin and 1% zinc. Alloy C consists of 80% copper, 15% tin and 5% zinc. How many kilograms of each of these alloys must be melted and mixed in order to produce 100 kg of a new alloy that consists of 87% copper, 9.6% tin and 3.4% zinc?