Section 1.1 INTRODUCTION TO PROBLEM SOLVING

There is no more significant privilege than to release the creative power of a child's mind.
Franz F. Hohn

PROBLEM OPENER

Alice counted 7 cycle riders and 19 cycle wheels going past her house. How many tricycles were there?

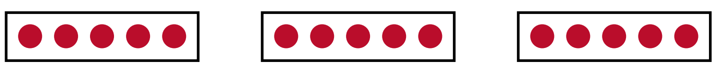
NCTM Standards

Problem solving is the hallmark of mathematical activity and a major means of developing mathematical knowledge. p. 116

"Learning to solve problems is the principal reason for studying mathematics."* This statement by the National Council of Supervisors of Mathematics represents a widespread opinion that problem solving should be the central focus of the mathematics curriculum.

A **problem** exists when there is a situation you want to resolve but no solution is readily apparent. **Problem solving** is the process by which the unfamiliar situation is resolved. A situation that is a problem to one person may not be a problem to someone else. For example, determining the number of people in 3 cars when each car contains 5 people may be a problem to some elementary school students. They might solve this problem by placing chips in boxes or by making a drawing to represent each car and each person (Figure 1.1) and then counting to determine the total number of people.

Figure 1.1



You may be surprised to know that some problems in mathematics are unsolved and have resisted the efforts of some of the best mathematicians to solve them. One such

*National Council of Supervisors of Mathematics, *Essential Mathematics for the 21st Century* (Minneapolis, MN: Essential Mathematics Task Force, 1988).

NCTM Standards

Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery. Teachers and researchers agree that students can learn to make, refine, and test conjectures in elementary school. p. 57

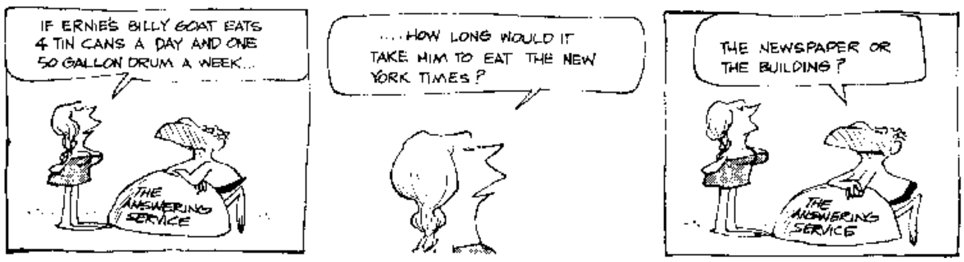
problem was discovered by Arthur Hamann, a seventh-grade student. He noticed that every even number could be written as the difference of two primes.* For example,

$$2 = 5 - 3 \qquad 4 = 11 - 7 \qquad 6 = 11 - 5 \qquad 8 = 13 - 5 \qquad 10 = 13 - 3$$

After showing that this was true for all even numbers less than 250, he predicted that every even number could be written as the difference of two primes. No one has been able to prove or disprove this statement. When a statement is thought to be true but remains unproved, it is called a **conjecture.**

Problem solving is the subject of a major portion of research and publishing in mathematics education. Much of this research is founded on the problem-solving writings of George Polya, one of the foremost twentieth-century mathematicians. Polya devoted much of his teaching to helping students become better problem solvers. His book *How to Solve It* has been translated into 18 languages. In this book, he outlines the following four-step process for solving problems.

Understanding the Problem Polya suggests that a problem solver needs to become better acquainted with a problem and work toward a clearer understanding of it before progressing toward a solution. Increased understanding can come from rereading the statement of the problem, drawing a sketch or diagram to show connections and relationships, restating the problem in your own words, or making a reasonable guess at the solution to help become acquainted with the details.



Sometimes the main difficulty in solving a problem is knowing what question is to be answered.

Devising a Plan The path from understanding a problem to devising a plan may sometimes be long. Most interesting problems do not have obvious solutions. Experience and practice are the best teachers for devising plans. Throughout the text you will be introduced to strategies for devising plans to solve problems.

Carrying Out the Plan The plan gives a general outline of direction. Write down your thinking so your steps can be retraced. Is it clear that each step has been done correctly? Also, it's all right to be stuck, and if this happens, it is sometimes better to put aside the problem and return to it later.

Looking Back When a result has been reached, verify or check it by referring to the original problem. In the process of reaching a solution, other ways of looking at the problem may become apparent. Quite often after you become familiar with a problem, new or perhaps more novel approaches may occur to you. Also, while solving a problem, you may find other interesting questions or variations that are worth exploring.

*M. R. Frame, "Hamann's Conjecture," *Arithmetic Teacher* 23, no. 1 (January 1976): 34–35.

I will learn to draw a diagram to solve problems.

24.3

Problem Solving: Strategy

Draw a Diagram

Read

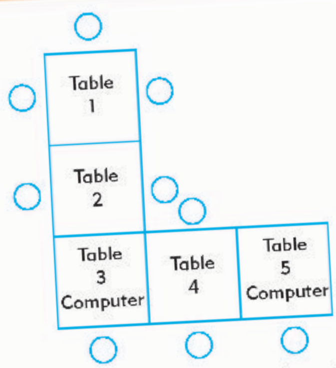
The two new computers for your classroom have finally arrived! You can help design the layout for the new computer center in your room. There are 5 tables available. At least 8 students have to be able to sit in the center at a time. Only 1 person can sit at a table with a computer. All the tables must be touching at least one other table. If one person can sit on each side of the other tables, how would you design the space?

- **What do you know?**
  5 tables available; must seat at least 8 students; 1 person at a table with a computer
- **What do you need to find?**
  How to arrange the tables

Plan

To solve, draw a diagram to represent the tables and chairs. Use squares for tables and circles for chairs.

Solve



Using the arrangement in this diagram, there are 9 seats available.

Look Back

How could you solve this problem in a different way?
Sample answer: If the computer is moved from table 5 to table 2 you would have room for 10 students

1. **Write About It** **Explain** how making a diagram helps you solve the problem.
Sample answer: The diagram helps me see different ways the tables and chairs can be arranged.

Polya's problem-solving steps will be used throughout the text. The purpose of this section is to help you become familiar with the four-step process and to acquaint you with some of the common strategies for solving problems: *making a drawing, guessing and checking, making a table, using a model,* and *working backward.* Additional strategies will be introduced throughout the text.

MAKING A DRAWING

One of the most helpful strategies for understanding a problem and obtaining ideas for a solution is to *draw sketches and diagrams.* Most likely you have heard the expression "A picture is worth a thousand words." In the following problem, the drawings will help you to think through the solution.

Problem

For his wife's birthday, Mr. Jones is planning a dinner party in a large recreation room. There will be 22 people, and in order to seat them he needs to borrow card tables, the size that seats one person on each side. He wants to arrange the tables in a rectangular shape so that they will look like one large table. What is the smallest number of tables that Mr. Jones needs to borrow?

NCTM Standards

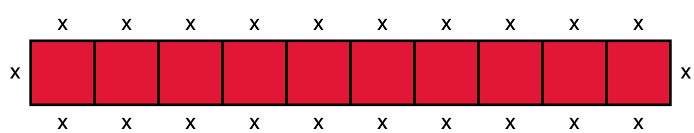
Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Polya (1957). Frequently cited strategies include using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backward, guessing and checking, creating an equivalent problem, and creating a simpler problem. p. 53

Understanding the Problem The tables must be placed next to each other, edge to edge, so that they form one large rectangular table. **Question 1:** If two tables are placed end to end, how many people can be seated?

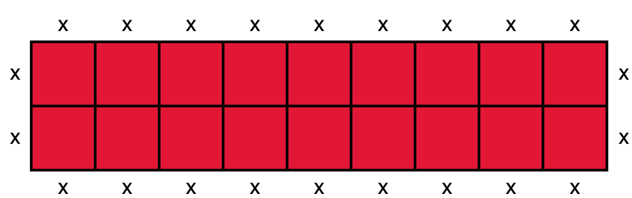
One large table

Devising a Plan Drawing pictures of the different arrangements of card tables is a natural approach to solving this problem. There are only a few possibilities. The tables can be placed in one long row; they can be placed side by side with two abreast; etc. **Question 2:** How many people can be seated at five tables if they are placed end to end in a single row?

Carrying Out the Plan The following drawings show two of the five possible arrangements that will seat 22 people. The X's show that 22 people can be seated in each arrangement. The remaining arrangements—3 by 8, 4 by 7, and 5 by 6—require 24, 28, and 30 card tables, respectively. **Question 3:** What is the smallest number of card tables needed?



10 tables



18 tables

Looking Back The drawings show that a single row of tables requires the fewest tables because each end table has places for 3 people and each of the remaining tables has places for 2 people. In all the other arrangements, the corner tables seat only 2 people and the remaining tables seat only 1 person. Therefore, regardless of the number of people, a single row is the arrangement that uses the smallest number of card tables, provided the room is long enough. **Question 4:** What is the smallest number of card tables required to seat 38 people?

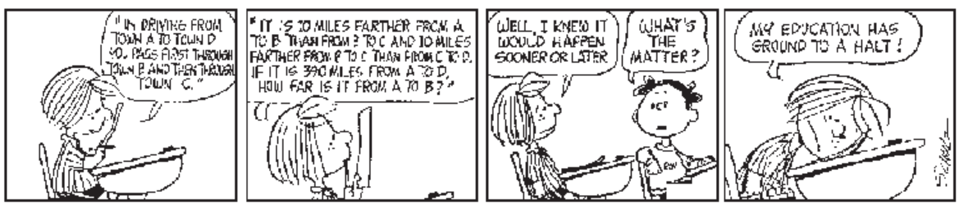
Answers to Questions 1–4 **1.** 6 **2.** 12 **3.** 10 **4.** There will be 3 people at each end table and 32 people in between. Therefore, 2 end tables and 16 tables in between will be needed to seat 38 people.

GUESSING AND CHECKING

Sometimes it doesn't pay to guess, as illustrated by the bus driver in this cartoon. However, many problems can be better understood and even solved by trial-and-error procedures. As Polya said, "Mathematics in the making consists of guesses." If your first guess is off, it may lead to a better guess. Even if guessing doesn't produce the correct answer, you may increase your understanding of the problem and obtain an idea for solving it. The *guess-and-check* approach is especially appropriate for elementary schoolchildren because it puts many problems within their reach.

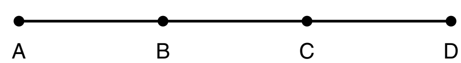
Problem

How far is it from town A to town B in this cartoon?

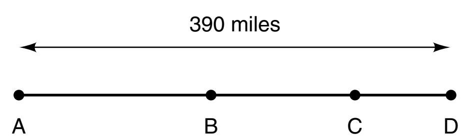


Peanuts: © United Feature Syndicate, Inc.

Understanding the Problem There are several bits of information in this problem. Let's see how Peppermint Patty could have obtained a better understanding of the problem with a diagram. First, let us assume these towns lie in a straight line, so they can be illustrated by points A, B, C, and D, as shown in (a). Next, it is 10 miles farther from A to B than from B to C, so we can move point B closer to point C, as in (b). It is also 10 miles farther from B to C than from C to D, so point C can be moved closer to point D. Finally, the distance from A to D is given as 390 miles. **Question 1:** The problem requires finding what distance?



(a)



(b)

NCTM Standards

Problem solving is not a distinct topic, but a process that should permeate the study of mathematics and provide a context in which concepts and skills are learned. p. 182

Devising a Plan One method of solving this problem is to make a reasonable guess and then use the result to make a better guess. If the 4 towns were equally spaced, as in (a), the distance between each town would be 130 miles (390 ÷ 3). However, the distance from town *A* to town *B* is the greatest. So let's begin with a guess of 150 miles for the distance from *A* to *B*. **Question 2:** In this case, what is the distance from *B* to *C* and *C* to *D*?

Carrying Out the Plan Using a guess of 150 for the distance from *A* to *B* produces a total distance from *A* to *D* that is greater than 390. If the distance from *A* to *B* is 145, then the *B*-to-*C* distance is 135 and the *C*-to-*D* distance is 125. The sum of these distances is 405, which is still too great. **Question 3:** What happens if we use a guess of 140 for the distance from *A* to *B*?

Looking Back One of the reasons for *looking back* at a problem is to consider different solutions or approaches. For example, you might have noticed that the first guess, which produced a distance of 420 miles, was 30 miles too great. **Question 4:** How can this observation be used to lead quickly to a correct solution of the original problem?

Answers to Questions 1–4 **1.** The problem requires finding the distance from *A* to *B*. **2.** The *B*-to-*C* distance is 140, and the *C*-to-*D* distance is 130. **3.** If the *A*-to-*B* distance is 140, then the *B*-to-*C* distance is 130 and the *C*-to-*D* distance is 120. Since the total of these distances is 390, the correct distance from *A* to *B* is 140 miles. **4.** If the distance between each of the 3 towns is decreased by 10 miles, the incorrect distance of 420 will be decreased to the correct distance of 390. Therefore, the distance between town *A* and town *B* is 140 miles.

MAKING A TABLE

A problem can sometimes be solved by listing some of or all the possibilities. A *table* is often convenient for organizing such a list.

Problem

Sue and Ann earned the same amount of money, although one worked 6 days more than the other. If Sue earned $36 per day and Ann earned $60 per day, how many days did each work?

Understanding the Problem Answer a few simple questions to get a feeling for the problem. **Question 1:** How much did Sue earn in 3 days? Did Sue earn as much in 3 days as Ann did in 2 days? Who worked more days?

Devising a Plan One method of solving this problem is to list each day and each person's total earnings through that day. **Question 2:** What is the first amount of total pay that is the same for Sue and Ann, and how many days did it take each to earn this amount?

Carrying Out the Plan The complete table is shown on page 9. There are three amounts in Sue's column that equal amounts in Ann's column. It took Sue 15 days to earn $540. **Question 3:** How many days did it take Ann to earn $540, and what is the difference between the numbers of days they each required?

Technology Connection

Four-Digit Numbers
If any four-digit number is selected and its digits reversed, will the sum of these two numbers be divisible by 11? Use your calculator to explore this and similar questions in this investigation.

Mathematics Investigation
Chapter 1, Section 1
www.mhhe.com/bennett-nelson

Number of Days	Sue's Pay	Ann's Pay
1	36	60
2	72	120
3	108	180
4	144	240
5	180	300
6	216	360
7	252	420
8	288	480
9	324	540
10	360	600
11	396	660
12	432	720
13	468	780
14	504	840
15	540	900

Looking Back You may have noticed that every 5 days Sue earns \$180 and every 3 days Ann earns \$180. **Question 4:** How does this observation suggest a different way to answer the original question?

Answers to Questions 1–4 **1.** Sue earned \$108 in 3 days. Sue did not earn as much in 3 days as Ann did in 2 days. Sue must have worked more days than Ann to have earned the same amount. **2.** \$180. It took Sue 5 days to earn \$180, and it took Ann 3 days to earn \$180. **3.** It took Ann 9 days to earn \$540, and the difference between the numbers of days Sue and Ann worked is 6. **4.** When Sue has worked 10 days and Ann has worked 6 days (a difference of 4 days), each has earned \$360; when they have worked 15 days and 9 days (a difference of 6 days), respectively, each has earned \$540.

USING A MODEL

Models are important aids for visualizing a problem and suggesting a solution. The recommendations by the Committee on the Undergraduate Program in Mathematics (CUPM) contain frequent references to the use of models for illustrating number relationships and geometric properties.*

The next problem uses **whole numbers** 0, 1, 2, 3, . . . and is solved by *using a model.* It involves a well-known story about the German mathematician Karl Gauss. When Gauss was 10 years old, his schoolmaster gave him the problem of computing the sum of whole numbers from 1 to 100. Within a few moments the young Gauss wrote the answer on his slate and passed it to the teacher. Before you read the solution to the following problem, try to find a quick method for computing the sum of whole numbers from 1 to 100.

*Committee on the Undergraduate Program in Mathematics, *Recommendations on the Mathematical Preparation of Teachers* (Berkeley, CA: Mathematical Association of America, 1983).

Problem

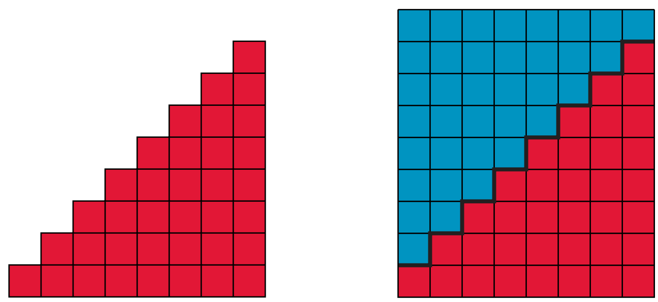
Find an easy method for computing the sum of consecutive whole numbers from 1 to any given number.

Understanding the Problem If the last number in the sum is 8, then the sum is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. If the last number in the sum is 100, then the sum is $1 + 2 + 3 + \cdots + 100$. **Question 1:** What is the sum of whole numbers from 1 to 8?

Devising a Plan One method of solving this problem is to cut staircases out of graph paper. The one shown in (a) is a 1-through-8 staircase: There is 1 square in the first step, there are 2 squares in the second step, and so forth, to the last step, which has a column of 8 squares. The total number of squares is the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. By using two copies of a staircase and placing them together, as in (b), we can obtain a rectangle whose total number of squares can easily be found by multiplying length by width. **Question 2:** What are the dimensions of the rectangle in (b), and how many small squares does it contain?

NCTM Standards

Problem solving is not only a goal of learning mathematics but also a major means of doing so. p. 52



1-through-8 staircase

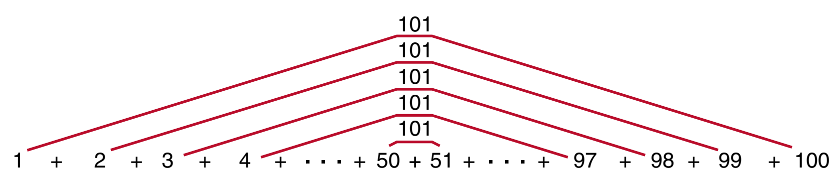
(a)

Two 1-through-8 staircases

(b)

Carrying Out the Plan Cut out two copies of the 1-through-8 staircase and place them together to form a rectangle. Since the total number of squares is 8×9, the number of squares in one of these staircases is $(8 \times 9)/2 = 36$. So the sum of whole numbers from 1 to 8 is 36. By placing two staircases together to form a rectangle, we see that the number of squares in one staircase is just half the number of squares in the rectangle. This geometric approach to the problem suggests that the sum of consecutive whole numbers from 1 to any specific number is the product of the last number and the next number, divided by 2. **Question 3:** What is the sum of whole numbers from 1 to 100?

Looking Back Another approach to computing the sum of whole numbers from 1 to 100 is suggested by the following diagram, and it may have been the method used by Gauss. If the numbers from 1 to 100 are paired as shown, the sum of each pair of numbers is 101.



Question 4: How can this sum be used to obtain the sum of whole numbers from 1 to 100?

Answers to Questions 1–4 **1.** 36 **2.** The dimensions are 8 by 9, and there are $8 \times 9 = 72$ small squares. **3.** Think of combining two 1-through-100 staircases to obtain a rectangle with 100×101 squares. The sum of whole numbers from 1 to 100 is $100(101)/2 = 5050$. **4.** Since there are 50 pairs of numbers and the sum for each pair is 101, the sum of numbers from 1 to 100 is $50 \times 101 = 5050$.

Hypatia, 370–415

HISTORICAL HIGHLIGHT

Athenaeus, a Greek writer (ca. 200), in his book *Deipnosophistoe* mentions a number of women who were superior mathematicians. However, Hypatia in the fourth century is the first woman in mathematics of whom we have considerable knowledge. Her father, Theon, was a professor of mathematics at the University of Alexandria and was influential in her intellectual development, which eventually surpassed his own. She became a student of Athens at the school conducted by Plutarch the Younger, and it was there that her fame as a mathematician became established. Upon her return to Alexandria, she accepted an invitation to teach mathematics at the university. Her contemporaries wrote about her great genius. Socrates, the historian, wrote that her home as well as her lecture room was frequented by the most unrelenting scholars of the day. Hypatia was the author of several treatises on mathematics, but only fragments of her work remain. A portion of her original treatise *On the Astronomical Canon of Diophantus* was found during the fifteenth century in the Vatican library. She also wrote *On the Conics of Apollonius.* She invented an astrolabe and a planesphere, both devices for studying astronomy, and apparatuses for distilling water and determining the specific gravity of water.*

*L. M. Osen, *Women in Mathematics* (Cambridge, MA: MIT Press, 1974), pp. 21–32.

WORKING BACKWARD

Problem

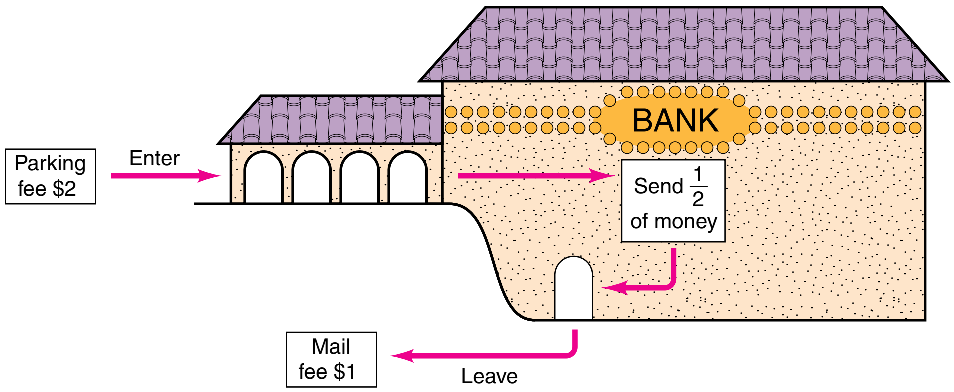
A businesswoman went to the bank and sent half of her money to a stockbroker. Other than a $2 parking fee before she entered the bank and a $1 mail fee after she left the bank, this was all the money she spent. On the second day she returned to the bank and sent half of her remaining money to the stockbroker. Once again, the only other expenses were the $2 parking fee and the $1 mail fee. If she had $182 left, how much money did she have before the trip to the bank on the first day?

Understanding the Problem Let's begin by guessing the original amount of money, say, $800, to get a better feel for the problem. **Question 1:** If the businesswoman begins the day with $800, how much money will she have at the end of the first day, after paying the mail fee?

Devising a Plan Guessing the original amount of money is one possible strategy, but it requires too many computations. Since we know the businesswoman has $182 at the end of the second day, a more appropriate strategy for solving the problem is to retrace her steps back through the bank (see the following diagram). First she receives $1 back from the mail fee. Continue to work back through the second day in the bank. **Question 2:** How much money did the businesswoman have at the beginning of the second day?

NCTM Standards

The goal of school mathematics should be for all students to become increasingly able and willing to engage with and solve problems. p. 182



Carrying Out the Plan The businesswoman had $368 at the beginning of the second day. Continue to work backward through the first day to determine how much money she had at the beginning of that day. **Question 3:** What was this amount?

Looking Back You can now check the solution by beginning with $740, the original amount of money, and going through the expenditures for both days to see if $182 is the remaining amount. The problem can be varied by replacing $182 at the end of the second day by any amount and working backward to the beginning of the first day. **Question 4:** For example, if there was $240 at the end of the second day, what was the original amount of money?

Answers to Questions 1–4 **1.** $398 **2.** The following diagram shows that the businesswoman had $368 at the beginning of the second day.

$$\underset{\text{End of day 2}}{\$182} \xrightarrow[\text{mail fee}]{\text{Receive \$1}} \$183 \xrightarrow[\text{money sent}]{\text{Receive 1/2 of}} \$366 \xrightarrow[\text{parking fee}]{\text{Receive \$2}} \underset{\text{Beginning of day 2}}{\$368}$$

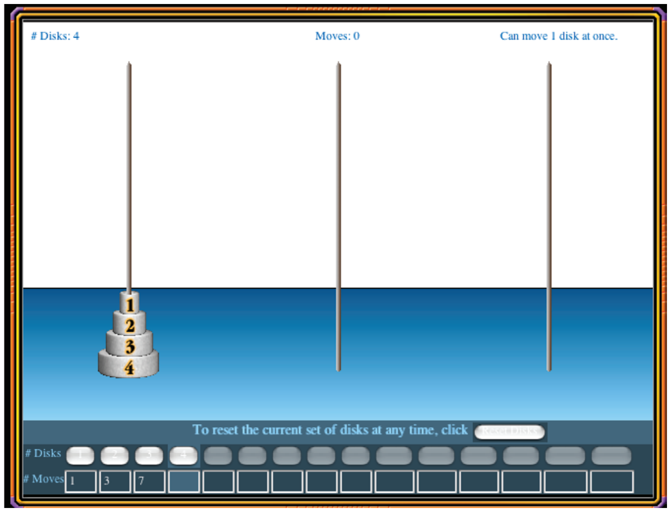
3. The diagram shows that the businesswoman had $740 at the beginning of the first day, so this is the original amount of money.

$$\underset{\text{End of day 1}}{\$368} \xrightarrow[\text{mail fee}]{\text{Receive \$1}} \$369 \xrightarrow[\text{money sent}]{\text{Receive 1/2 of}} \$738 \xrightarrow[\text{parking fee}]{\text{Receive \$2}} \underset{\text{Beginning of day 1}}{\$740}$$

4. $972

Technology Connection

What is the least number of moves to transfer four disks from one tower to another if only one disk can be moved at a time and a disk cannot be placed on top of a smaller disk? In this applet, you will solve an ancient problem by finding patterns to determine the minimum number of moves for transferring an arbitrary number of disks.



Tower Puzzle Patterns Applet, Chapter 1
www.mhhe.com/bennett-nelson

Exercises and Problems 1.1

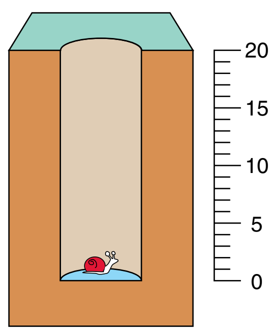
Problems 1 through 20 involve strategies that were presented in this section. Some of these problems are analyzed by Polya's four-step process. See if you can solve these problems before answering parts a, b, c, and d. Other strategies may occur to you, and you are encouraged to use the ones you wish. Often a good problem requires several strategies.

Making a Drawing (1–4)

PS **1.** A well is 20 feet deep. A snail at the bottom climbs up 4 feet each day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?

a. Understanding the Problem What is the greatest height the snail reaches during the first 24 hours? How far up the well will the snail be at the end of the first 24 hours?

b. Devising a Plan One plan that is commonly chosen is to compute $20/2$, since it appears that the snail gains 2 feet each day. However, 10 days is not the correct answer. A second plan is to *make a drawing* and plot the snail's daily progress. What is the snail's greatest height during the second day?



c. Carrying Out the Plan Trace out the snail's daily progress, and mark its position at the end of each day. On which day does the snail get out of the well?

d. Looking Back There is a "surprise ending" at the top of the well because the snail does not slip back on the ninth day. Make up a new snail problem by changing the numbers so that there will be a similar surprise ending at the top of the well.

2. Five people enter a racquetball tournament in which each person must play every other person exactly once. Determine the total number of games that will be played.

3. When two pieces of rope are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

4. There are 560 third- and fourth-grade students in King Elementary School. If there are 80 more third-graders than fourth-graders, how many third-graders are there in the school?

Making a Table (5–8)

5. A bank that has been charging a monthly service fee of \$2 for checking accounts plus 15 cents for each check announces that it will change its monthly fee to \$3 and that each check will cost 8 cents. The bank claims the new plan will save the customer money. How many checks must a customer write per month before the new plan is cheaper than the old plan?

a. Understanding the Problem Try some numbers to get a feel for the problem. Compute the cost of 10 checks under the old plan and under the new plan. Which plan is cheaper for a customer who writes 10 checks per month?

b. Devising a Plan One method of solving this problem is to make a table showing the cost of 1 check, 2 checks, etc., such as that shown here. How much more does the new plan cost than the old plan for 6 checks?

Checks	Cost for Old Plan, \$	Cost for New Plan, \$
1	2.15	3.08
2	2.30	3.16
3	2.45	3.24
4	2.60	3.32
5	2.75	3.40
6		
7		
8		

c. Carrying Out the Plan Extend the table until you reach a point at which the new plan is cheaper than the old plan. How many checks must be written per month for the new plan to be cheaper?

d. Looking Back For customers who write 1 check per month, the difference in cost between the old plan and the new plan is 93 cents. What happens to the difference as the number of checks increases? How many checks must a customer write per month before the new plan is 33 cents cheaper?

6. Sasha and Francisco were selling lemonade for 25 cents per half cup and 50 cents per full cup. At the end of the day they had collected \$15 and had used 37 cups. How many full cups and how many half cups did they sell?

7. Harold wrote to 15 people, and the cost of postage was \$4.71. If it cost 23 cents to mail a postcard and 37 cents to mail a letter, how many postcards did he write?

8. I had some pennies, nickels, dimes, and quarters in my pocket. When I reached in and pulled out some change, I had less than 10 coins whose value was 42 cents. What are all the possibilities for the coins I had in my hand?

Guessing and Checking (9–12)

9. There are two 2-digit numbers that satisfy the following conditions: (1) Each number has the same digits, (2) the sum of the digits in each number is 10, and (3) the difference between the 2 numbers is 54. What are the two numbers?

a. Understanding the Problem The numbers 58 and 85 are 2-digit numbers that have the same digits, and the sum of the digits in each number is 13. Find two 2-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.

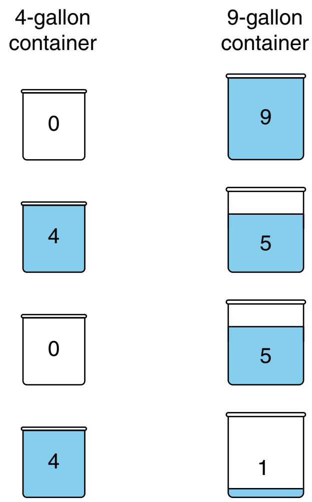
b. Devising a Plan Since there are only nine 2-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two 2-digit numbers from part a? If this difference is not 54, it can provide information about your next guess.

c. Carrying Out the Plan Continue to guess and check. Which pair of numbers has a difference of 54?

d. Looking Back This problem can be extended by changing the requirement that the sum of the two digits equals 10. Solve the problem for the case in which the digits have a sum of 12.

10. When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are these two numbers?

11. When asked how a person can measure out 1 gallon of water with only a 4-gallon container and a 9-gallon container, a student used this "picture."



a. Briefly describe what the student could have shown by this sketch.

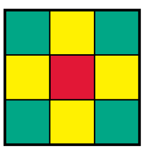
b. Use a similar sketch to show how 6 gallons can be measured out by using these same containers.

12. Carmela opened her piggy bank and found she had \$15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?

Using a Model (13–16)

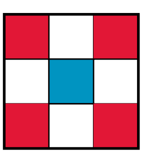
13. Suppose that you have a supply of red, blue, green, and yellow square tiles. What is the fewest number of different colors needed to form a 3 × 3 square of tiles so that no tile touches another tile of the same color at any point?

a. Understanding the Problem Why is the square arrangement of tiles shown here not a correct solution?



b. Devising a Plan One plan is to choose a tile for the center of the grid and then place others around it so that no two of the same color touch. Why must the center tile be a different color than the other eight tiles?

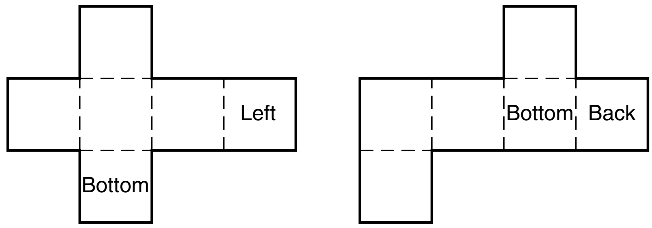
c. Carrying Out the Plan Suppose that you put a blue tile in the center and a red tile in each corner, as shown here. Why will it require two more colors for the remaining openings?



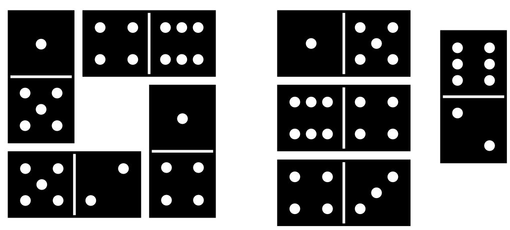
d. Looking Back Suppose the problem had asked for the smallest number of colors to form a square of nine tiles so that no tile touches another tile of the same color along an entire edge. Can it be done in fewer colors; if so, how many?

14. What is the smallest number of different colors of tile needed to form a 4 × 4 square so that no tile touches another of the same color along an entire edge?

15. The following patterns can be used to form a cube. A cube has six faces: the top and bottom faces, the left and right faces, and the front and back faces. Two faces have been labeled on each of the following patterns. Label the remaining four faces on each pattern so that when the cube is assembled with the labels on the outside, each face will be in the correct place.



16. At the left in the following figure is a domino doughnut with 11 dots on each side. Arrange the four single dominoes on the right into a domino doughnut so that all four sides have 12 dots.



Domino doughnut

Working Backward (17–20)

17. Three girls play three rounds of a game. On each round there are two winners and one loser. The girl who loses on a round has to double the number of chips that each of the other girls has by giving up some of her own chips. Each girl loses one round. At the end of three rounds, each girl has 40 chips. How many chips did each girl have at the beginning of the game?

a. Understanding the Problem Let's select some numbers to get a feel for this game. Suppose girl *A*, girl *B*, and girl *C* have 70, 30, and 20 chips, respectively, and girl A loses the first round. Girl B and girl C will receive chips from girl A, and thus their supply of chips will be doubled. How many chips will each girl have after this round?

b. Devising a Plan Since we know the end result (each girl finished with 40 chips), a natural strategy is to work backward through the three rounds to the beginning. Assume that girl C loses the third round. How many chips did each girl have at the end of the second round?

	A	*B*	*C*
Beginning			
End of first round			
End of second round			
End of third round	40	40	40

c. Carrying Out the Plan Assume that girl B loses the second round and girl A loses the first round. Continue working back through the three rounds to determine the number of chips each of the girls had at the beginning of the game.

d. Looking Back Check your answer by working forward from the beginning. The girl with the most chips at the beginning of this game lost the first round. Could the girl with the fewest chips at the beginning of the game have lost the first round? Try it.

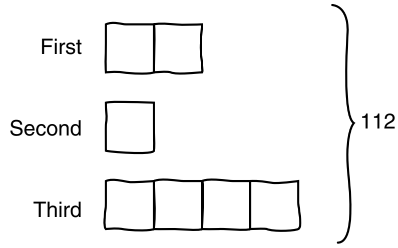
18. Sue Ellen and Angela both have \$510 in their savings accounts now. They opened their accounts on the same day, at which time Sue Ellen started with \$70 more than Angela. From then on Sue Ellen added \$10 to her account each week, and Angela put in \$20 each week. How much money did Sue Ellen open her account with?

19. Ramon took a collection of colored tiles from a box. Amelia took 13 tiles from his collection, and Keiko took half of those remaining. Ramon had 11 left. How many did he start with?

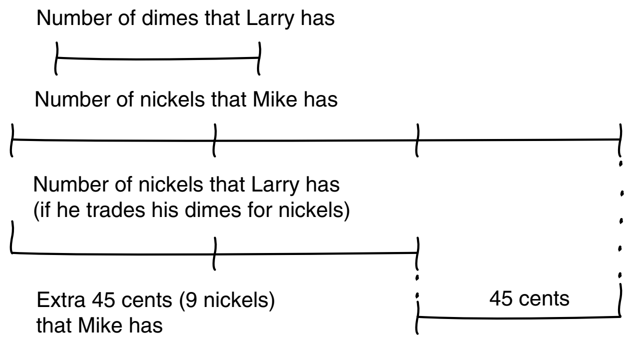
20. Keiko had 6 more red tiles than yellow tiles. She gave half of her red tiles to Amelia and half of her yellow tiles to Ramon. If Ramon has 7 yellow tiles, how many tiles does Keiko have?

Each of problems 21 to 24 is accompanied by a sketch or diagram that was used by a student to solve it. Describe how you think the student used the diagram, and use this method to solve the problem.

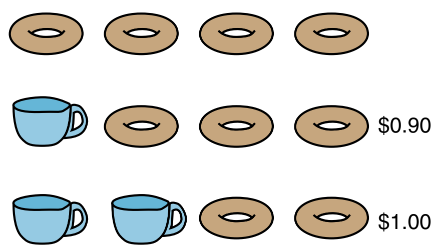
21. There are three numbers. The first number is twice the second number. The third is twice the first number. Their sum is 112. What are the numbers?



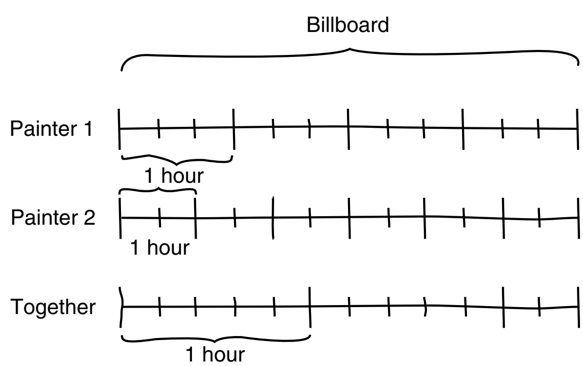
22. Mike has 3 times as many nickels as Larry has dimes. Mike has 45 cents more than Larry. How much money does Mike have?



23. At Joe's Cafe 1 cup of coffee and 3 doughnuts cost \$0.90, and 2 cups of coffee and 2 doughnuts cost \$1.00. What is the cost of 1 cup of coffee? 1 doughnut?



24. One painter can letter a billboard in 4 hours and another requires 6 hours. How long will it take them together to letter the billboard?



Problems 25 through 34 can be solved by using strategies presented in this section. While you are problem-solving, try to record the strategies you are using. If you are using a strategy different from those of this section, try to identify and record it.

25. There were ships with 3 masts and ships with 4 masts at the Tall Ships Exhibition. Millie counted a total of 30 masts on the 8 ships she saw. How many of these ships had 4 masts?

26. When a teacher counted her students in groups of 4, there were 2 students left over. When she counted them in groups of 5, she had 1 student left over. If 15 of her students were girls and she had more girls than boys, how many students did she have?

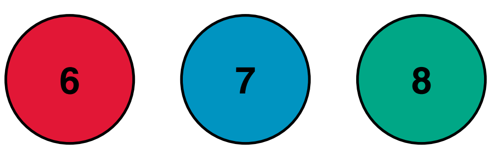
27. The video club to which Lin belongs allows her to receive a free movie video for every three videos she rents. If she pays \$3 for each movie video and paid \$132 over a 4-month period, how many free movie videos did she obtain?

28. Linda picked a basket of apples. She gave half of the apples to a neighbor, then 8 apples to her mother, then half of the remaining apples to her best friend, and she kept the 3 remaining apples for herself. How many apples did she start with in the basket?

29. Four people want to cross the river. There is only one boat available, and it can carry a maximum of 200 pounds. The weights of the four people are 190, 170, 110, and 90 pounds. How can they all manage to get across the river, and what is the minimum number of crossings required for the boat?

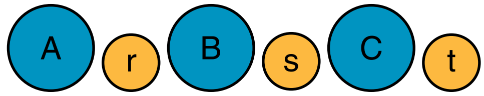
30. A farmer has to get a fox, a goose, and a bag of corn across a river in a boat that is only large enough for her and one of these three items. She does not want to leave the fox alone with the goose nor the goose alone with the corn. How can she get all these items across the river?

31. Three circular cardboard disks have numbers written on the front and back sides. The front sides have the numbers shown here.

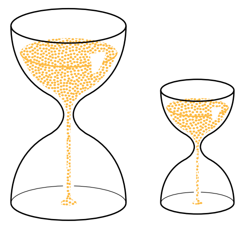


By tossing all three disks and adding the numbers that show face up, we can obtain these totals: 15, 16, 17, 18, 19, 20, 21, and 22. What numbers are written on the back sides of these disks?

32. By moving adjacent disks two at a time, you can change the arrangement of large and small disks shown below to an arrangement in which 3 big disks are side by side followed by the 3 little disks. Describe the steps.



33. How can a chef use an 11-minute hourglass and a 7-minute hourglass to time vegetables that must steam for 15 minutes?



34. The curator of an art exhibit wants to place security guards along the four walls of a large auditorium so that each wall has the same number of guards. Any guard who is placed in a corner can watch the two adjacent walls, but each of the other guards can watch only the wall by which she or he is placed.

a. Draw a sketch to show how this can be done with 6 security guards.

b. Show how this can be done for each of the following numbers of security guards: 7, 8, 9, 10, 11, and 12.

c. List all the numbers less than 100 that are solutions to this problem.

35. Trick questions like the following are fun, and they can help improve problem-solving ability because they require that a person listen and think carefully about the information and the question.

a. Take 2 apples from 3 apples, and what do you have?

b. A farmer had 17 sheep, and all but 9 died. How many sheep did he have left?

c. I have two U.S. coins that total 30 cents. One is not a nickel. What are the two coins?

d. A bottle of cider costs 86 cents. The cider costs 60 cents more than the bottle. How much does the bottle cost?

e. How much dirt is in a hole 3 feet long, 2 feet wide, and 2 feet deep?

f. A hen weighs 3 pounds plus half its weight. How much does it weigh?

g. There are nine brothers in a family and each brother has a sister. How many children are in the family?

h. Which of the following expressions is correct? (1) The whites of the egg are yellow. (2) The whites of the egg is yellow.

Writing and Discussion

1. Suppose one of your elementary school students was having trouble solving the following problem and asked for help: "Tauna gave half of her marbles away. If she gave some to her sister and twice as many to her brother, and had 6 marbles left, how many marbles did she give to her brother?" List a few suggestions you can give to this student to help her solve this problem.
2. When an elementary schoolteacher who had been teaching problem solving introduced the strategy of *making a drawing,* one of her students said that he was not good at drawing. Give examples of three problems you can give this student that would illustrate that artistic ability is not required. Accompany the problems with solution sketches.
3. In years past, it was a common practice for teachers to tell students not to draw pictures or sketches because "you can't prove anything with drawings." Today it is common for teachers to encourage students to form sketches to solve problems. Discuss this change in approach to teaching mathematics. Give examples of advantages and disadvantages of solving problems by *making drawings.*
4. At one time, teachers scolded students for guessing the answers to problems. In recent years, mathematics educators have recommended that *guessing and checking* be taught to school students. Write a few sentences to discuss the advantages of teaching students to "guess and check." Include examples of problems for which this strategy may be helpful.
5. Write a definition of what it means for a question to involve "problem solving." Create a problem that is appropriate for middle school students and explain how it satisfies your definition of problem solving.

Making Connections

1. The **Spotlight on Teaching** at the beginning of Chapter 1 poses the following problem: I have pennies, nickels, and dimes in my pocket. If I take three coins out of my pocket, how much money could I have taken? The solution in this spotlight involves *forming a table.* Explain and illustrate how a different organized list can lead to a solution by noting that the greatest value of the coins is 30 cents and the least value is 3 cents.
2. On page 5, the example from the **Elementary School Text** poses a problem and solves it by the strategy of *making a drawing.* Then it asks the students how the problem can be solved in a different way. (a) Find another solution that has seats for exactly 8 students. (b) Name a strategy from the **Standards** quote on page 6 that is helpful in solving part (a) and explain why the strategy is helpful.
3. The **Standards** quote on page 8 says that problem solving should "provide a context in which concepts and skills are learned." Explain how the staircase model, page 10, provides this context.
4. In the **Process Standard on Problem Solving** (see inside front cover), read the fourth expectation and explain several ways in which Polya's fourth problem-solving step addresses the fourth expectation.
5. The **Historical Highlight** on page 11 has some examples of the accomplishments of Hypatia, one of the first women mathematicians. Learn more about her by researching history of math books or searching the Internet. Record some interesting facts or anecdotes about Hypatia that you could use to enhance your elementary school teaching.
6. The **Problem Opener** for this section, on page 3, says that "Alice counted 7 cycle riders and 19 cycle wheels" and it asks for the number of tricycles. Use one or more of the problem-solving strategies in this section to find all the different answers that are possible if the riders might have been using unicycles, bicycles, or tricycles.